Folding of the Randomly Triangulated Surface : Review

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2003/12/16

- Folding of Geometric Objects
- 1-dim. Case : Meander Problem
- 2-dim. Case : Polymer(ized) Membrane
- 2-dim. Case : Fluid Membrane (under construction)

1 Folding of Geometric Objects

Geometric Objects ?

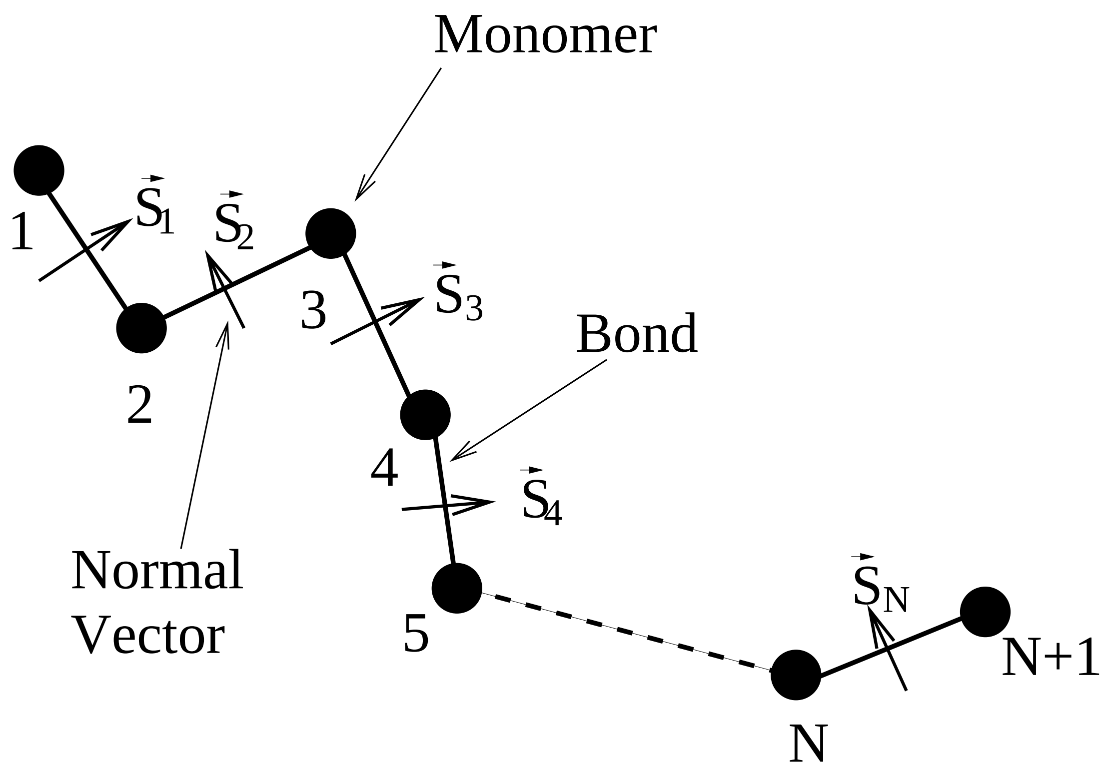
- 1-D string like objects
  Polymer, Flux line, Vortex line,
  World line (Trajectory of Particle)

- 2-D objects
  Fluid Membrane, Polymer Membrane,
  World Sheet (Trajectory of String)

- 3-D objects
  Gel

- 4-D objects
  Space-Time (Universe)

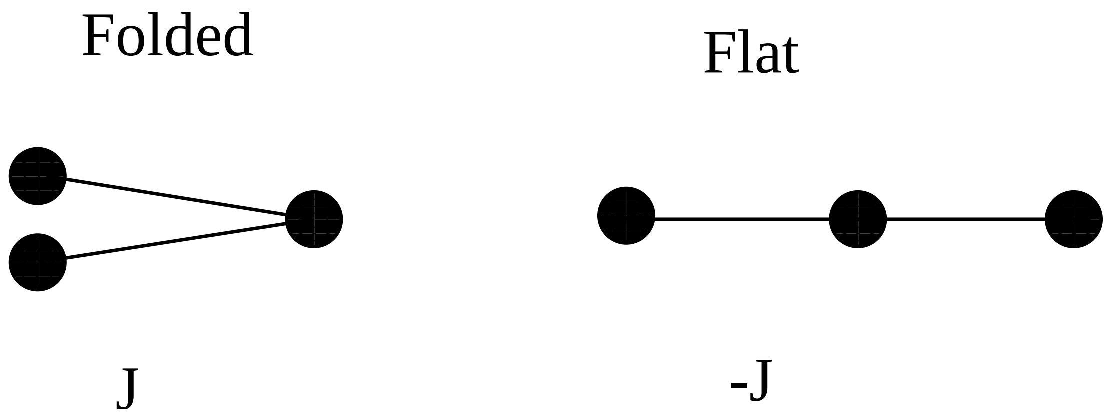
Polymer ?



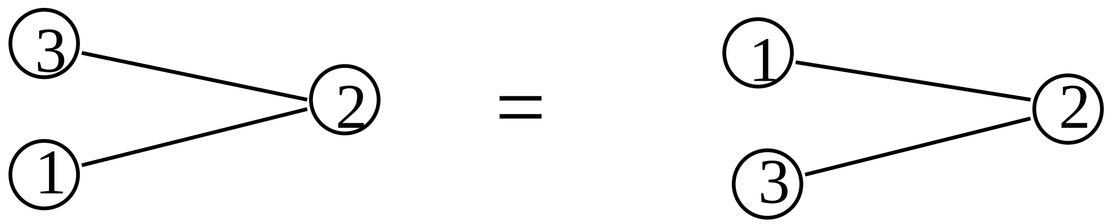
$$E = -J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \tag{1}$$

J : Bending Energy

Embedded in 1-dimensional Space → Fold or Flat



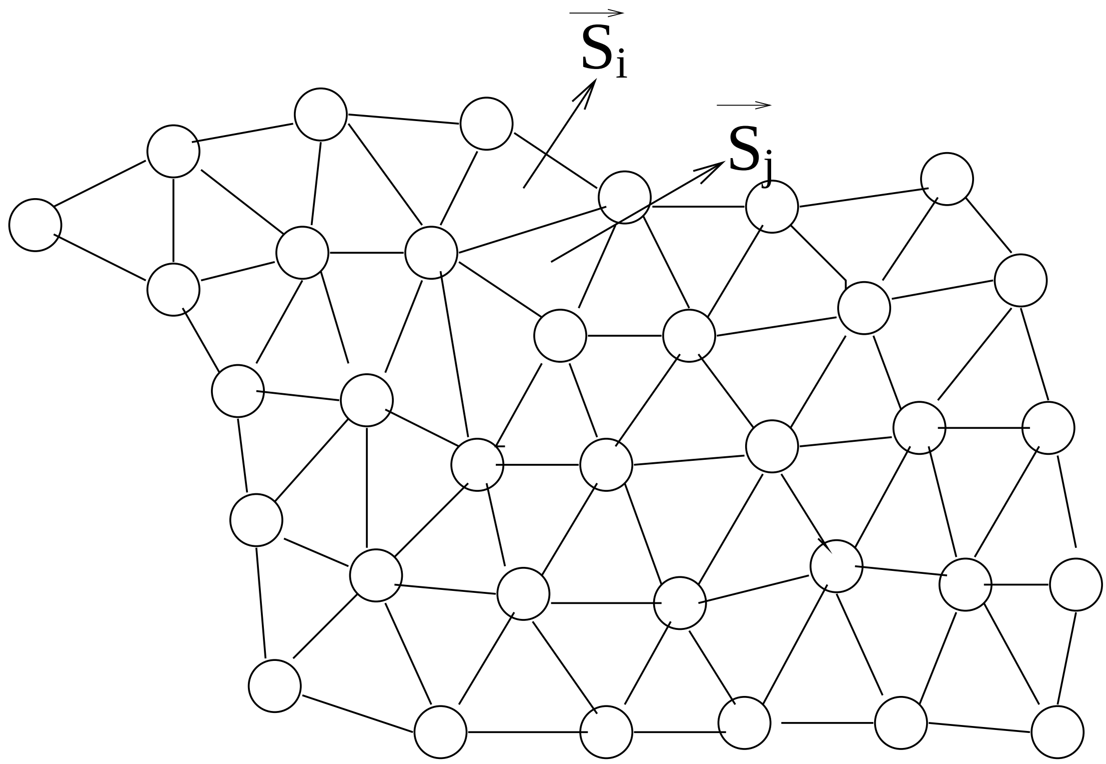
Phantom Case is very Simple !



$$E = -J \sum_{i=1}^{N} S_i S_{i+1} \tag{2}$$

Self-Avoiding Case is Difficult. → Meander Problem

Membrane ?



$$E = -J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \tag{3}$$

- Polymer(ized) Membrane

  Triangulated Surface with Fixed Connectivity

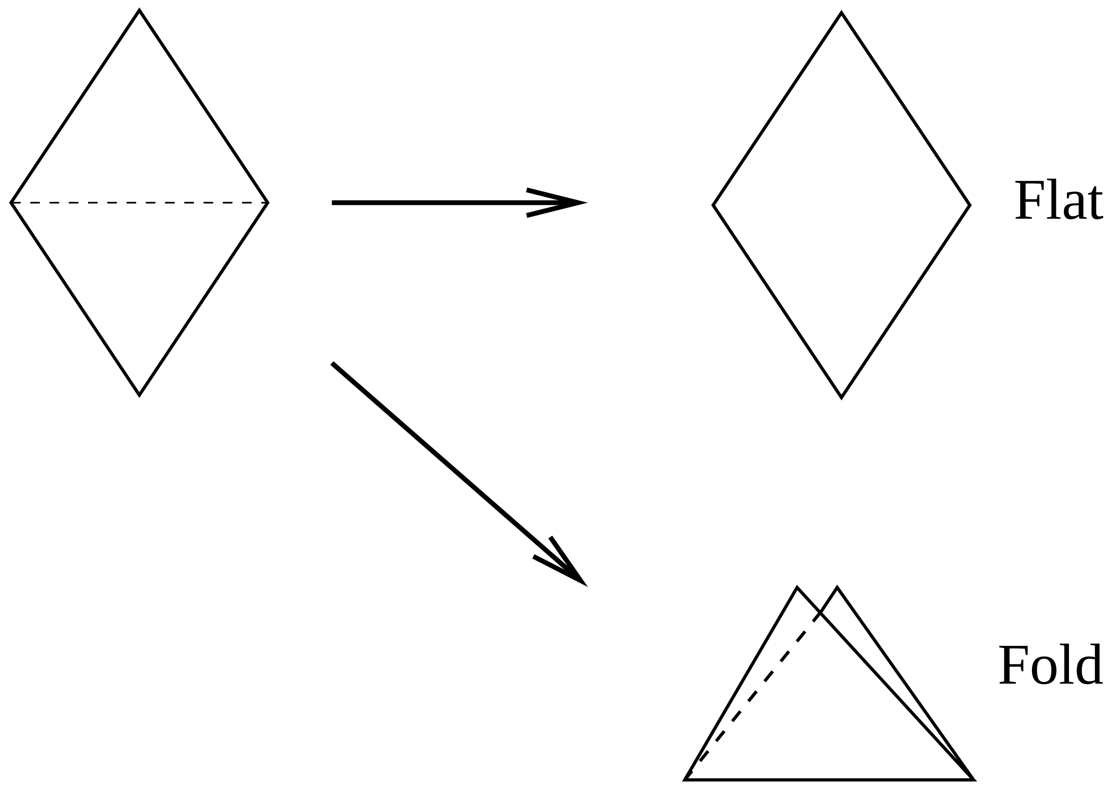
  - Regular Triangular Lattice
  - Randomly Polymerized Membrane (Quenched Randomly Triangulated Surface)

- Fluid Membrane

  Triangulated Surface with Random Connectivity

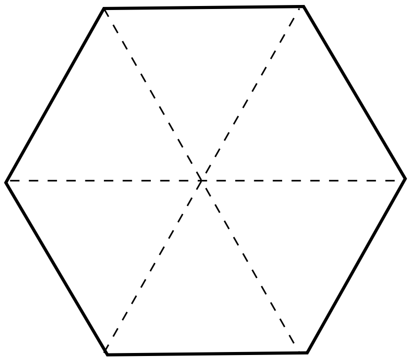
  Randomly Triangulated Surface

Embedded in 2-dimensional Space → Fold or Flat



Even Phantom Case is not Simple !

Folding of Elementary Hexagon:



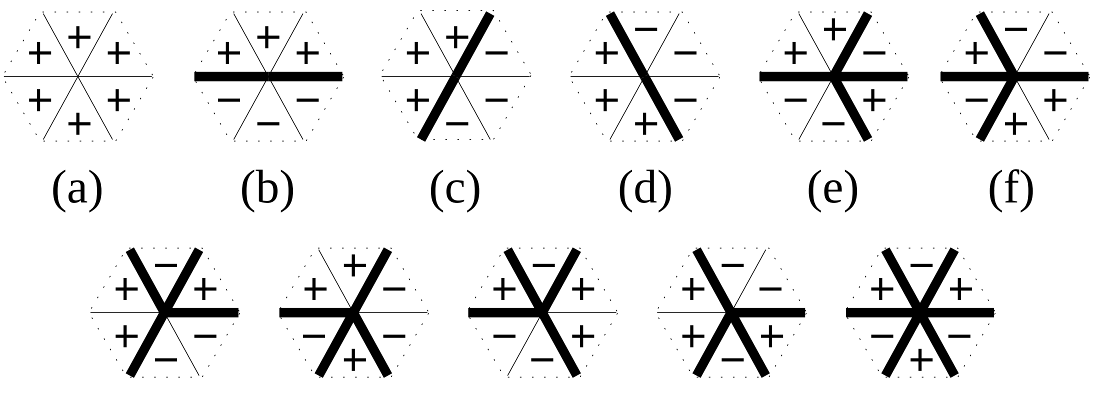
6 Bonds ----> 64 States ?

Only 11 Physical States

11 local fold environments for elementary Hexagon

Face Up $- - - - - - - - - - - - - - - - - > S = +1$

Face Down $- - - - - - - - - - - - - - - - > S = -1$

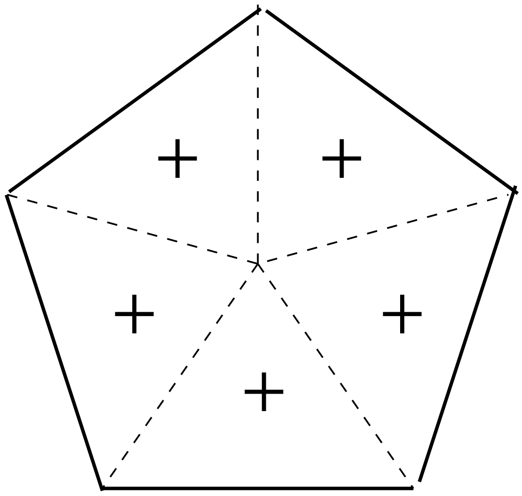


(a) (b) (c) (d) (e) (f)

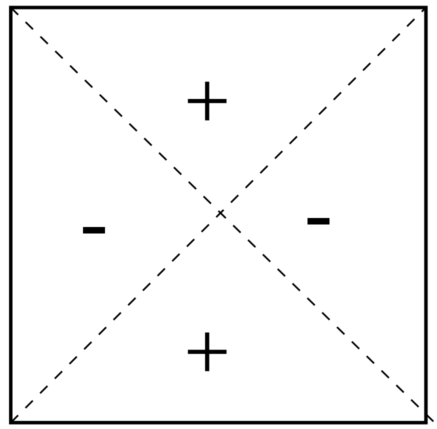
(g) (h) (i) (j) (k)

Randomly Triangulated Case:

Is It Foldable ?



Not Foldable



Foldable

Qustions :

- Foldability ?
- How to generate Foldable Randomly Triangulated Surface ?
- How to Enumerate ? Matrix Model, Decorated Tree ?

2 Meander Problem

2.1 Definition

(Semi-)Meander = Closed Non-Self-intersecting loop (road) crossing a semi-infinite line (river with a Pond) through n points (bridges).

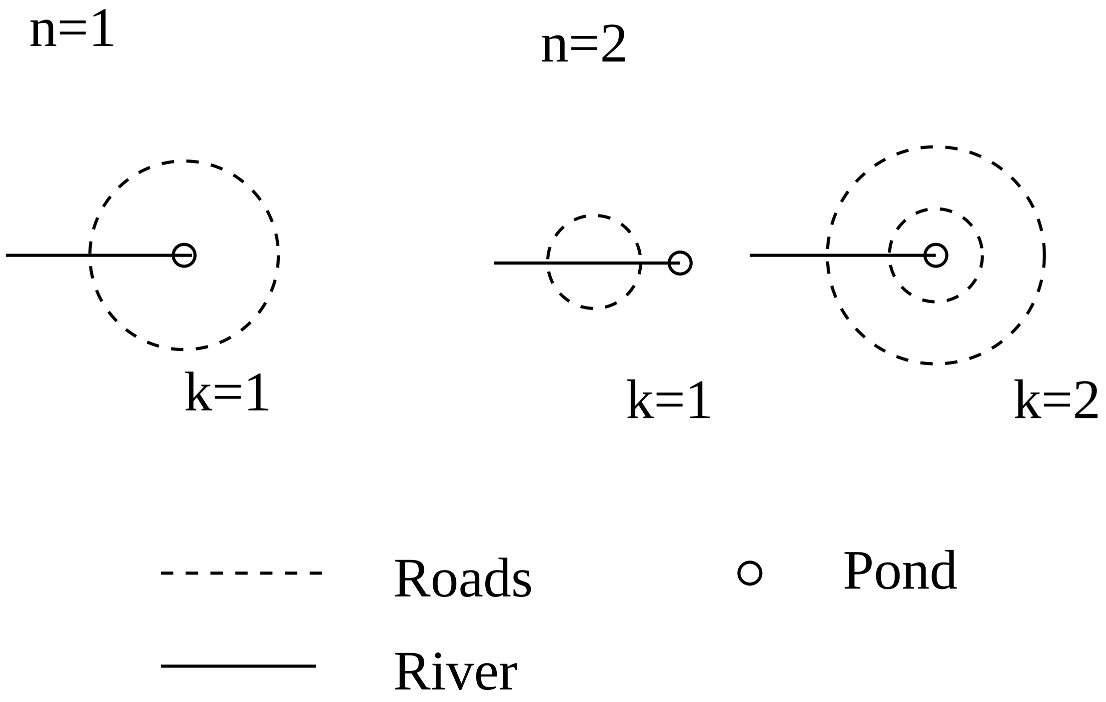


図 1: Meander with $n = 1, 2$. k is number of Connected Components.

$$M_1^1 = 1, \quad M_2^1 = 1, \quad M_2^2 = 1$$

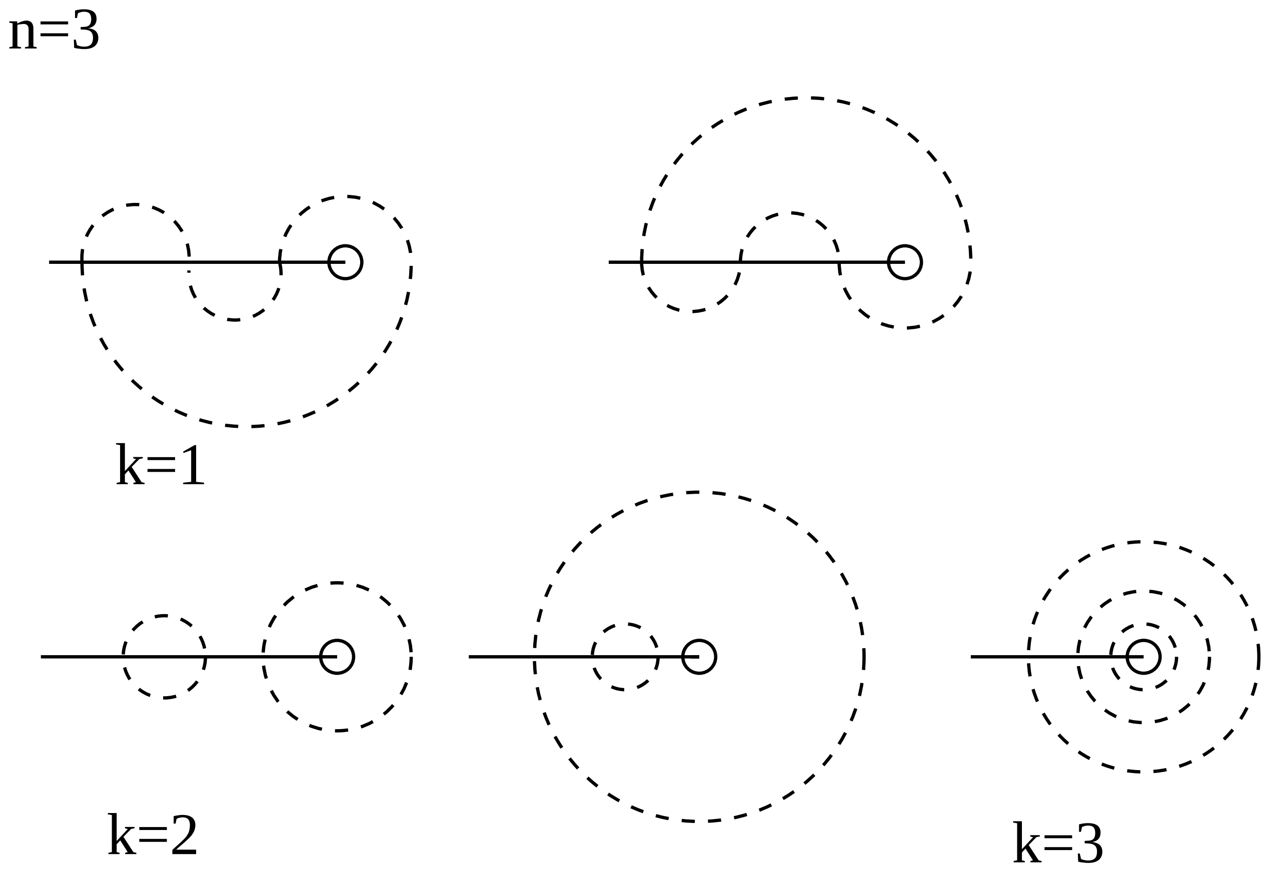


図 2: Meander with $n = 3$.

$$M_3^1 = 2, \quad M_3^2 = 2, \quad M_3^3 = 1$$

2.2 Compact Folding of Polymer

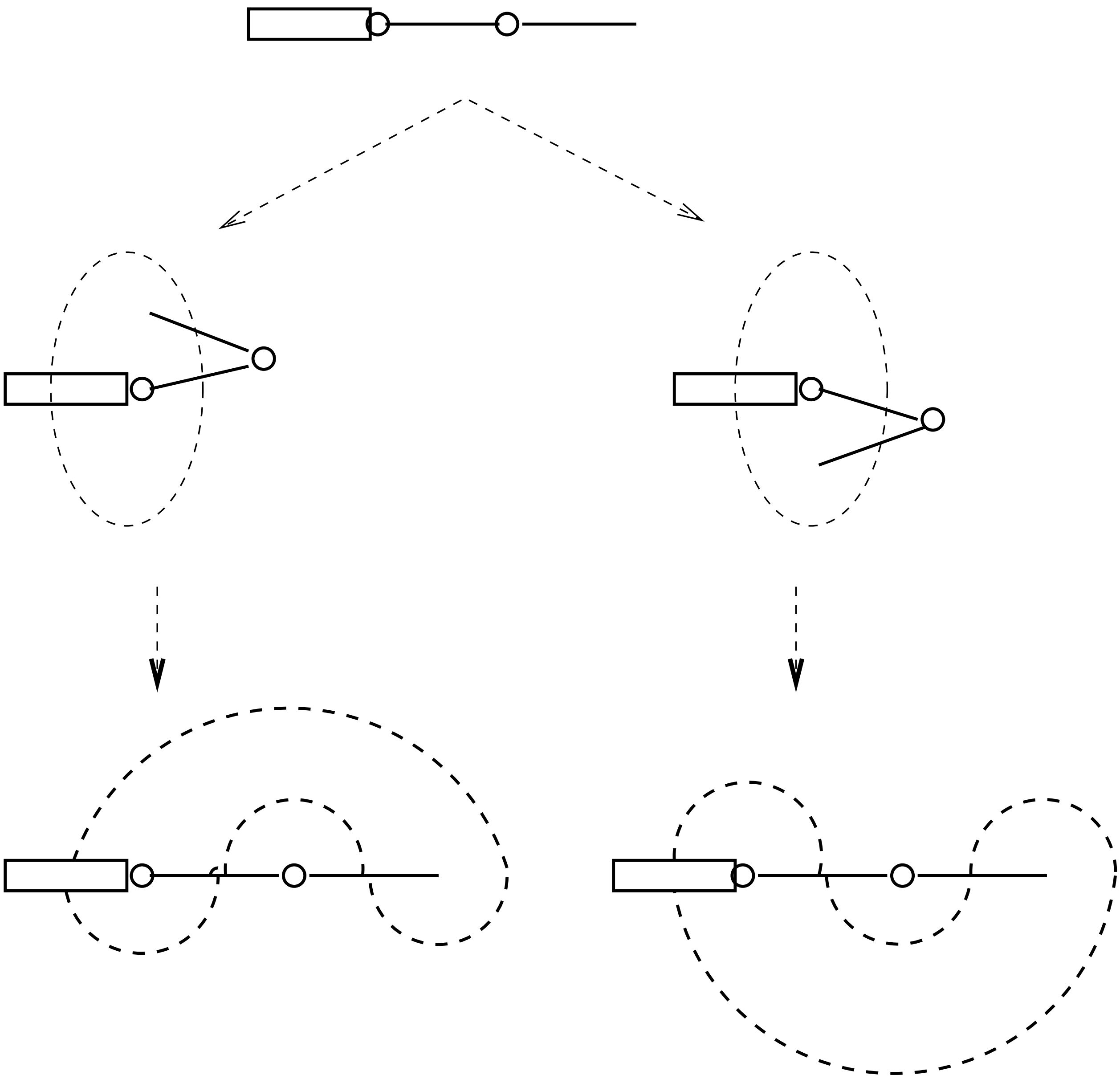


図 3: Meander and Compact Folding of Polymer

2.3 Folding Pathway of Polymer

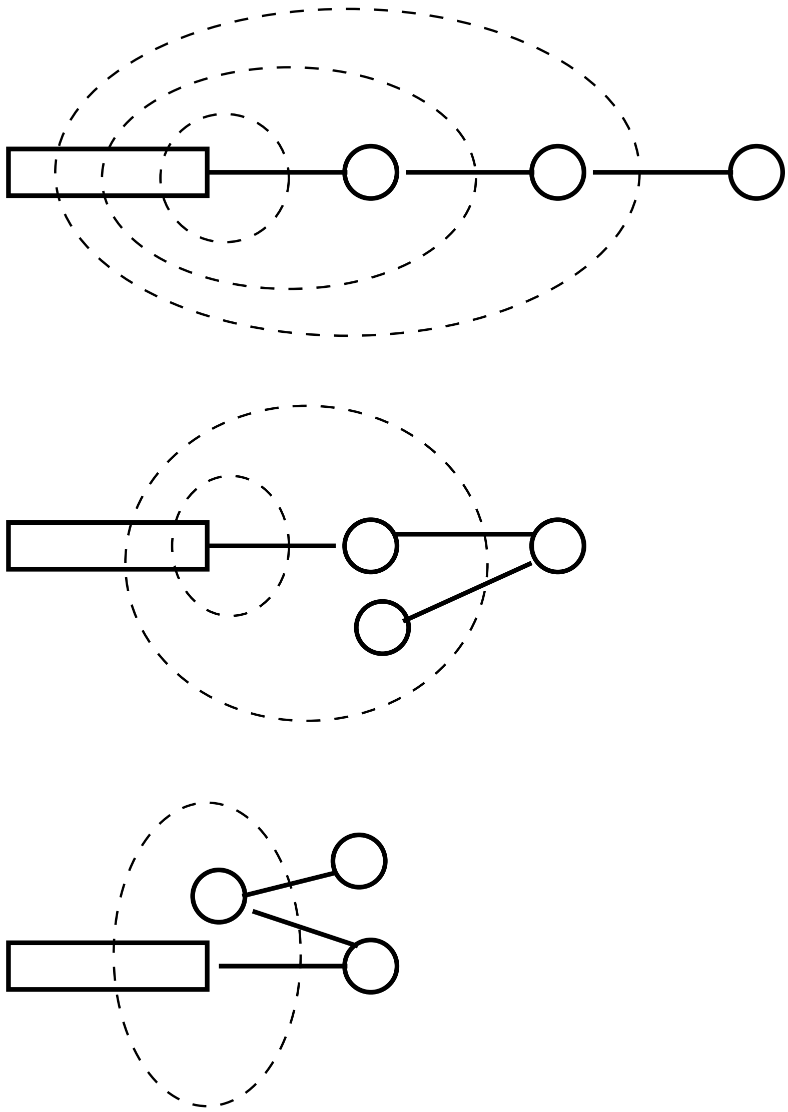


図 4: Folding Pathway and Meander

2.4 Meander Number: Numerical Data

表 1: The numbers M_n^k of inequivalent (semi)-meanders of order n with k connected components for $1 \leq n \leq 12$, obtained by exact enumeration on the computer.

k\ n	1	2	3	4	5	6	7	8	9
1	1	1	2	4	10	24	66	174	504
2	0	1	2	6	16	48	140	428	1308
3	0	0	1	3	11	37	126	430	1454
4	0	0	0	1	4	17	66	254	956
5	0	0	0	0	1	5	24	104	438
6	0	0	0	0	0	1	6	32	152
7	0	0	0	0	0	0	1	7	41
8	0	0	0	0	0	0	0	1	8
9	0	0	0	0	0	0	0	0	1

Enumerated Up to $n \leq 29$.

2.5 Properties of M_n^k

$$
\begin{aligned}
&M_n^n = 1 \\
&M_n^{n-1} = n - 1 \\
&\sum_{k=1}^{n} M_n^k = C_n
\end{aligned}
\tag{4}
$$

C_n: Catalan Number

$$
C_n \equiv \frac{1}{2n+1}\,{}_{2n+1}C_n = \frac{(2n)!}{n!(n+1)!} \simeq \frac{1}{\sqrt{\pi}}\frac{4^n}{n^{\frac{1}{2}}} \tag{5}
$$

$$
C_1 = 1 \qquad C_2 = 2, \qquad C_3 = 5, \qquad C_4 = 14 \tag{6}
$$

2.6 Arch and Catalan Number

$$C_n = \text{Number of Arch Configuration with } 2n \text{ bridges} \quad (7)$$

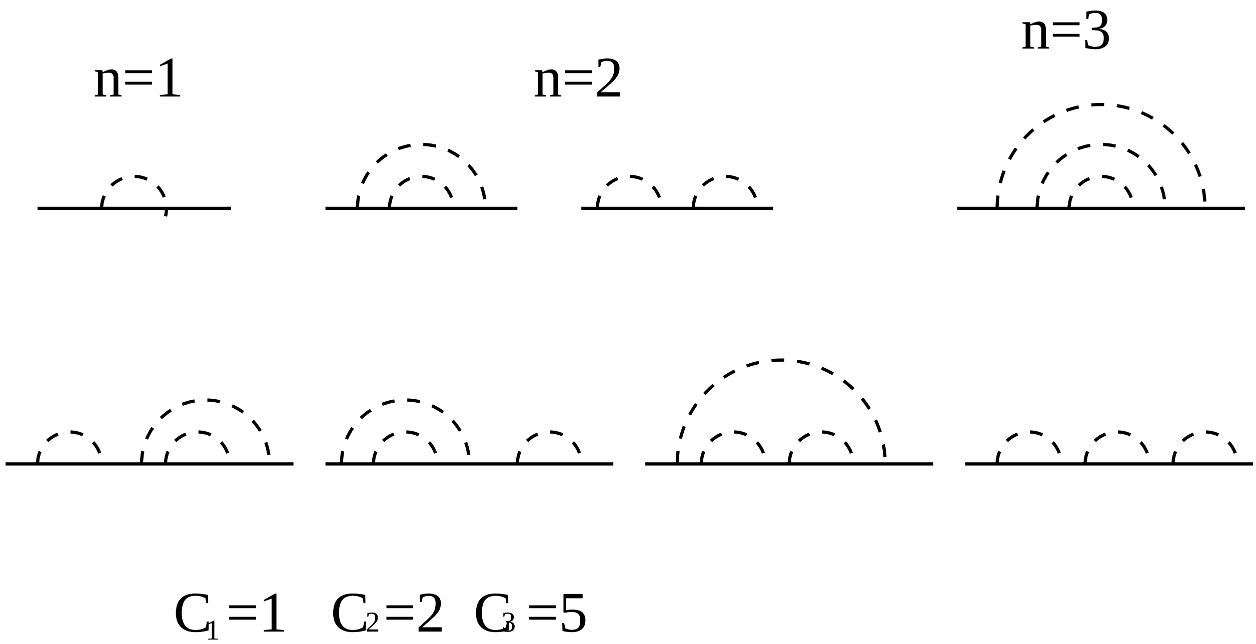


図 5: Arch and Catalan Number

2.7 Open-River Representation

$$\text{Meander} = \text{Arch} + \text{Rainbow} \tag{8}$$

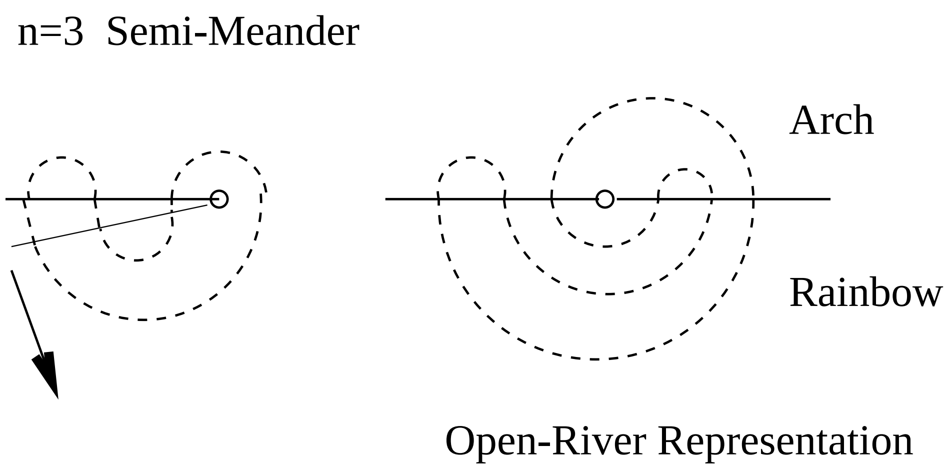


図 6: Arch + Rainbow =Meander

$$\sum_{k=1}^{n} M_n^k = C_n \tag{9}$$

2.8 Catalan Number C_n

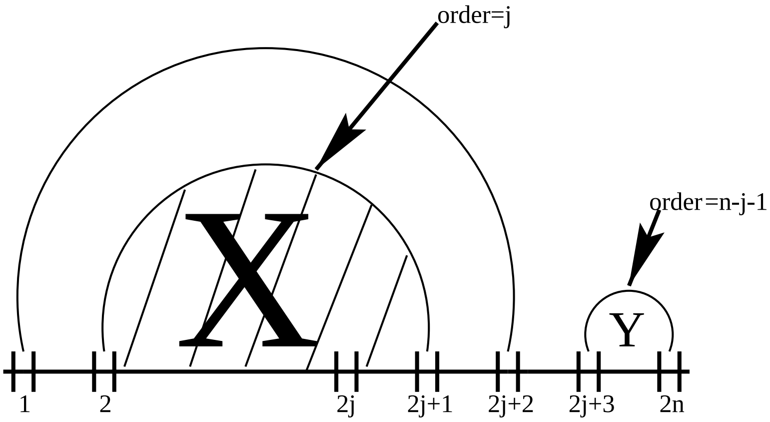


図 7:

$$C_n = \sum_{j=0}^{n-1} C_j \times C_{n-j-1} \quad \text{with} \quad C_0 = 1 \tag{10}$$

Generationg function $C(x)$:

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = 1 + C_1 x + C_2 x^2 + \cdots \tag{11}$$

$C(x)$ satisfy the following algebraic relation.

$$xC(x)^2 = C(x) - 1 \tag{12}$$

$$C(x) = \frac{1 - \sqrt{(1-4x)}}{2x} \tag{13}$$

2.9 How to Analyze

- Enumeration upto $n \leq 29$
  Ph.Di Francesco, O.Golinelli and E.Guitter

- Transfer-Matrix
  Ph. Di Francesco, O.Golinelli and E.Guitter

- Monte-Carlo Method (Population Sampling Method)
  $k = 1$ O.Golinelli, $k \geq 2$ S.Mori

- Mean-Field Method

- Temperley-Lieb Algebra
  Ph.Di Francesco, O.Golinelli and E.Guitter

- Matrix Model
  S.Land and A.Zvonkin, Ph.Di Francesco and E.Guitter

2.10 Numerical Methods

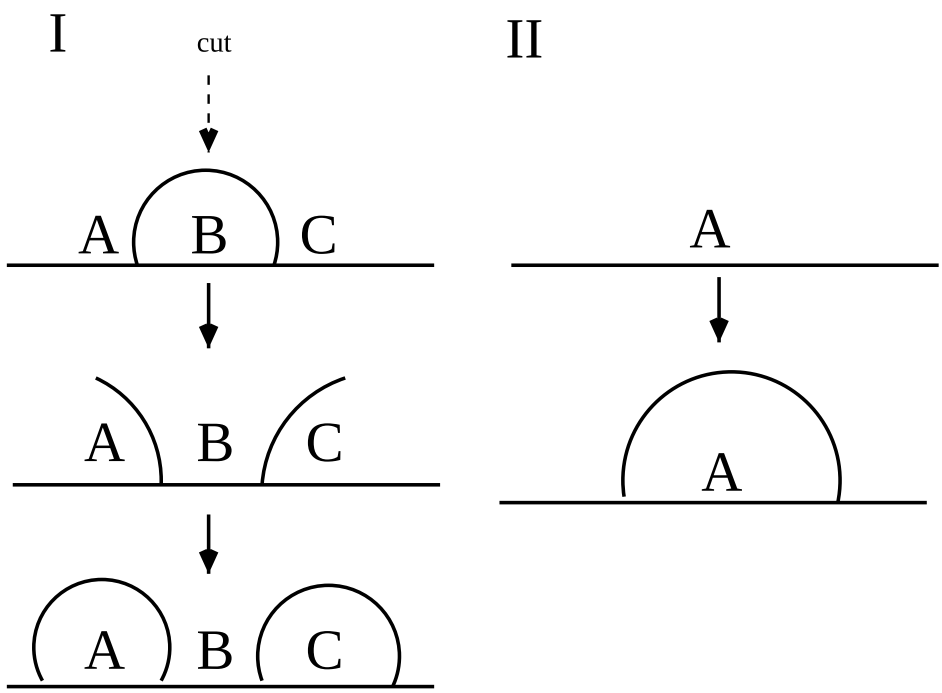


図 8: Construction of Arch : Process I and II

- Process I : Cut Exterior Arch and Pull its Ends all the way around the others and reconnect them below.

$$k \to k, n \to n+1$$

- Process II : Draw a Circle around the Meander.

$$k \to k+1, n \to n+1$$

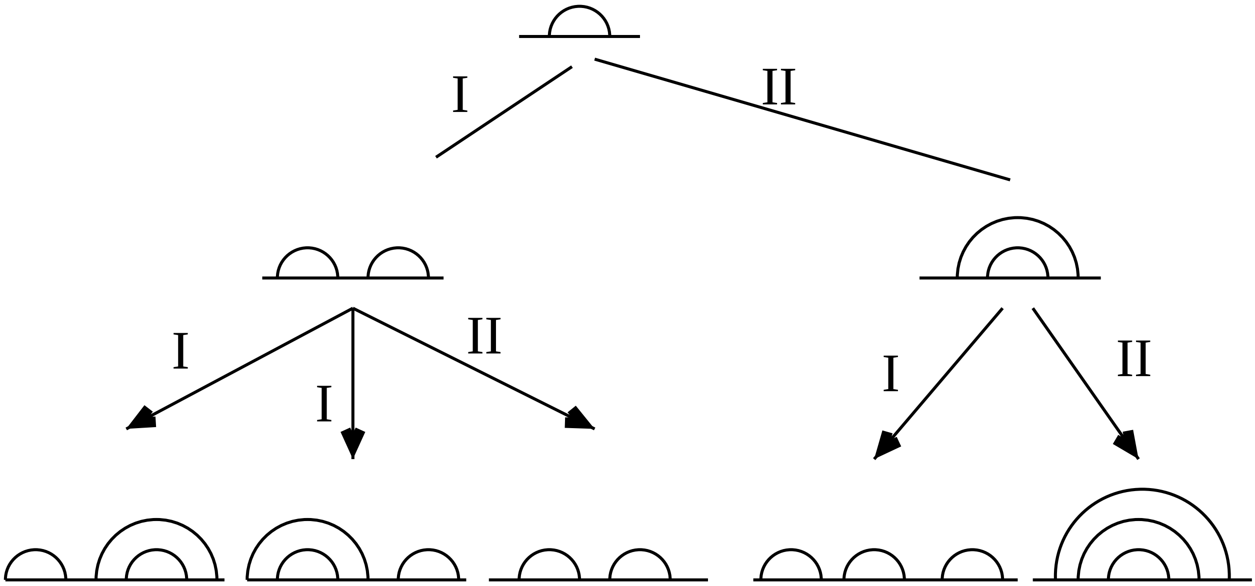


図 9: Arch $n \leq 3$

2.11 Asymptotic Behavior

From Enumeration, Monte-Carlo Method

O.Golinelli

$$M_n^k \sim \frac{R_k^n}{n^{\gamma_k}} \quad \text{with} \quad R_1 \simeq 3.5018(3) \quad \text{and} \quad \gamma_1 = 2.056(10) \tag{14}$$

S.Mori

表 2: γ_k

k	1	2	3	4	5
γ_k	2.0	1.4	0.45	-0.57	-1.53
k	6	7	8	9	10
γ_k	-2.53	-3.53	-4.53	-5.53	-6.6

表 3: R_k

k	1	2	3	4	5
$\log(R_k/3.5)$	0.0	0.005	0.003	0.001	0.001
k	6	7	8	9	10
$\log(R_k/3.5)$	0.0001	0.001	0.001	0.001	0.001

From Enumeration, Transfer Matrix, Matrix Model

Meander-Polynomial $m_n(q)$:

$$m_n(q) \equiv \sum_{k=1}^{n} M_n^k q^k \tag{15}$$

$$m_1(q) = q \qquad m_2(q) = q + q^2$$

$$m_3(q) = 2q + 2q^2 + q^3 \tag{16}$$

$$m_n(q) \sim \qquad \frac{R(q)^n}{n^{\gamma(q)}}$$

$$\text{with} \;\; R(1) = 4 \;\; \text{and} \;\; \gamma(1) = \frac{3}{2} \tag{17}$$

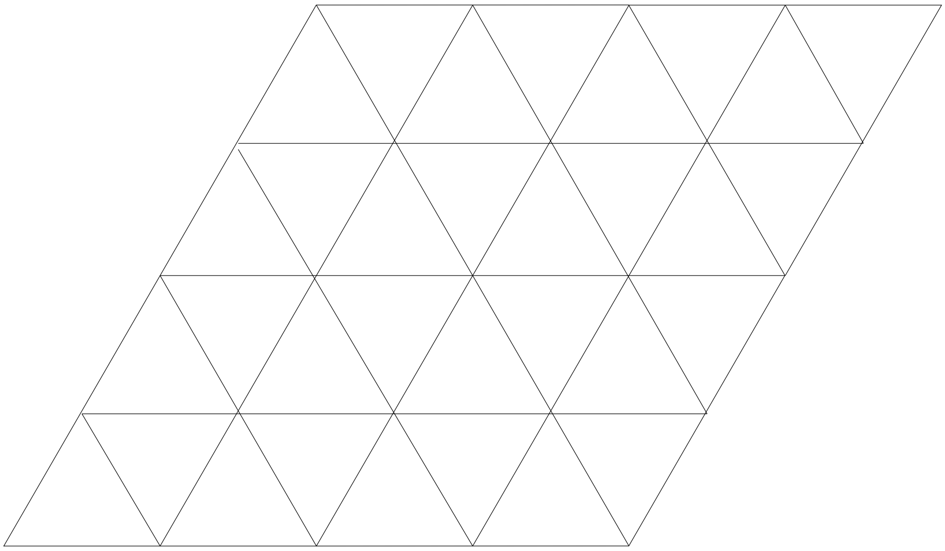
$$\text{as} \quad m_n(1) = C_n \simeq \frac{4^n}{n^{\frac{3}{2}}}$$

$$\text{with} \;\; R(\infty) = q \;\; \text{and} \;\; \gamma(\infty) = 0$$

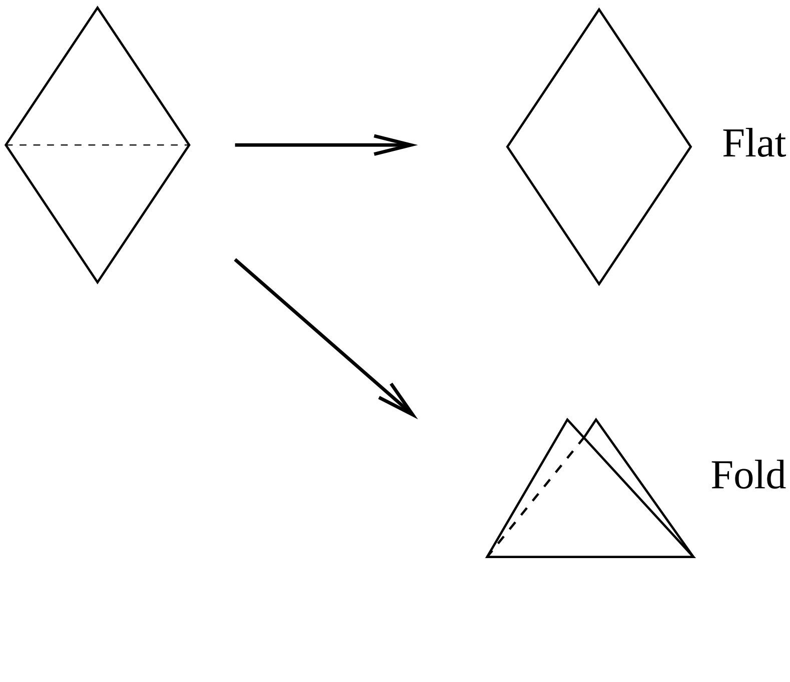
$$\text{as} \quad m_n(q) \simeq q^n \;\; \text{for} \;\; q \to \infty \tag{18}$$

3 Folding of the Regular Triangular Lattice

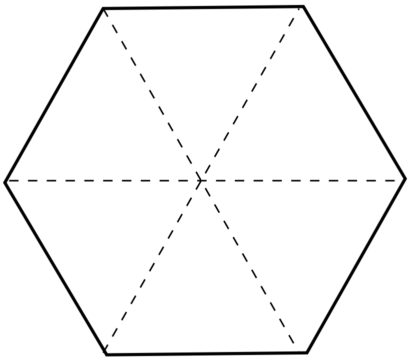
Regular Triangular Lattice



Embedded in 2-dimensional Space → Fold or Flat



Folding of Elementary Hexagon:



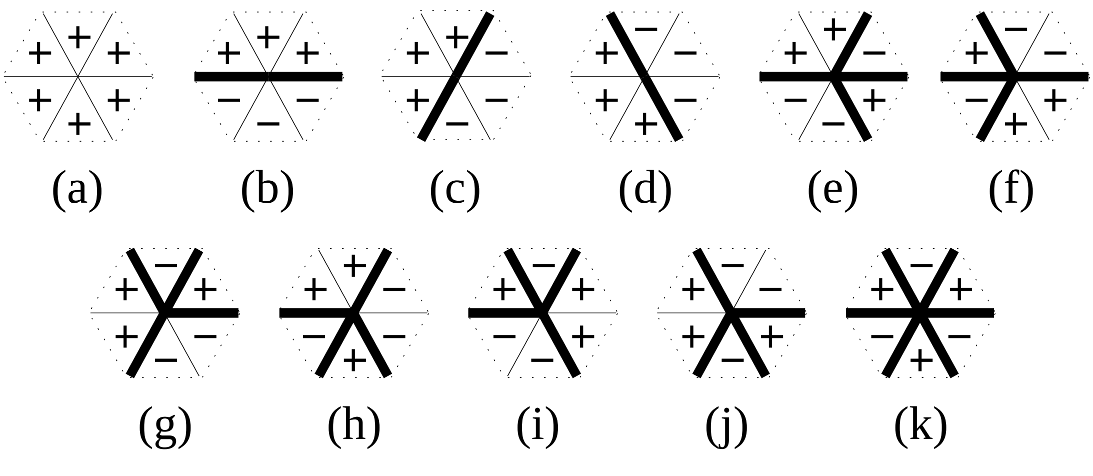
6 Bonds ----> 64 States ?

Only 11 Physical States

11 local fold environments for elementary Hexagon

Face Up $- - - - - - - - - - - - - - - - -> S = +1$

Face Down $- - - - - - - - - - - - - - - -> S = -1$



Model System:

Geometric Constraints :

$$\sigma_v \equiv \sum_{i \ \ around \ \ v} S_i = 0 \mod 3 \tag{19}$$

Constrained Z_2 Spin System : 11 vertex model

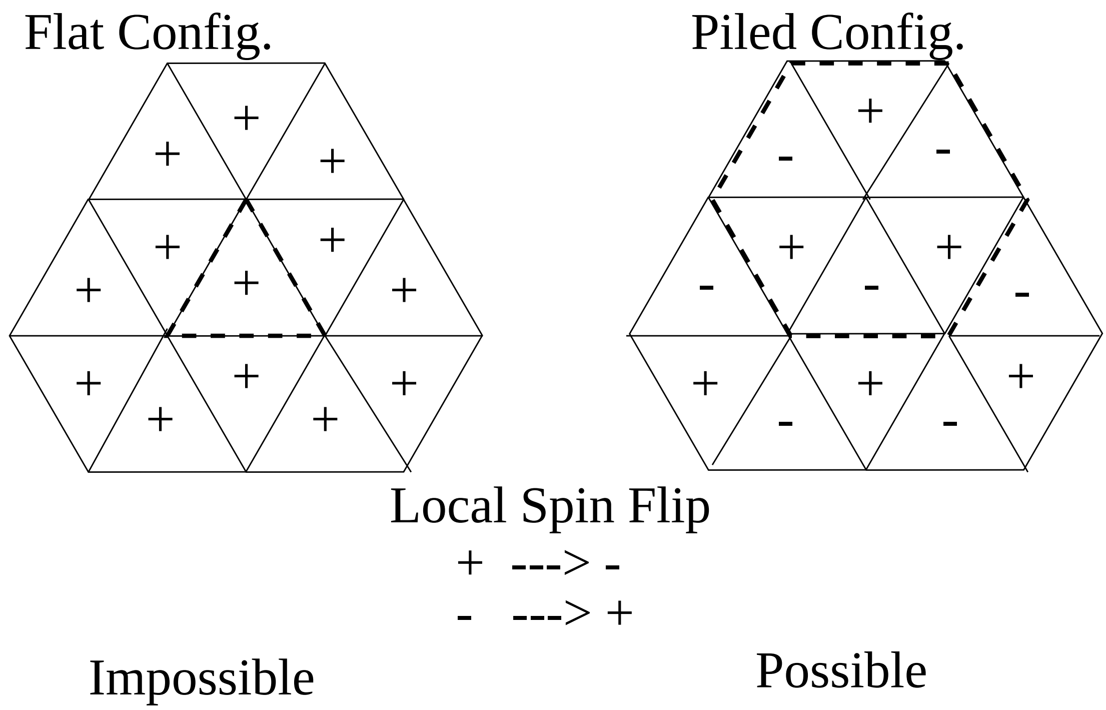
Hamiltonian:

$$\mathcal{H}_{Ising} = -J \sum_{<ij>} S_i S_j - H \sum_i S_i \tag{20}$$

and

$$K = \beta J = J/k_B T \text{and} h = \beta H \tag{21}$$

Structure of Phase Space: (Ph. Di Francesco and E.Guitter)



Local Spin Excitation :

(1) Impossible for Flat Configuration

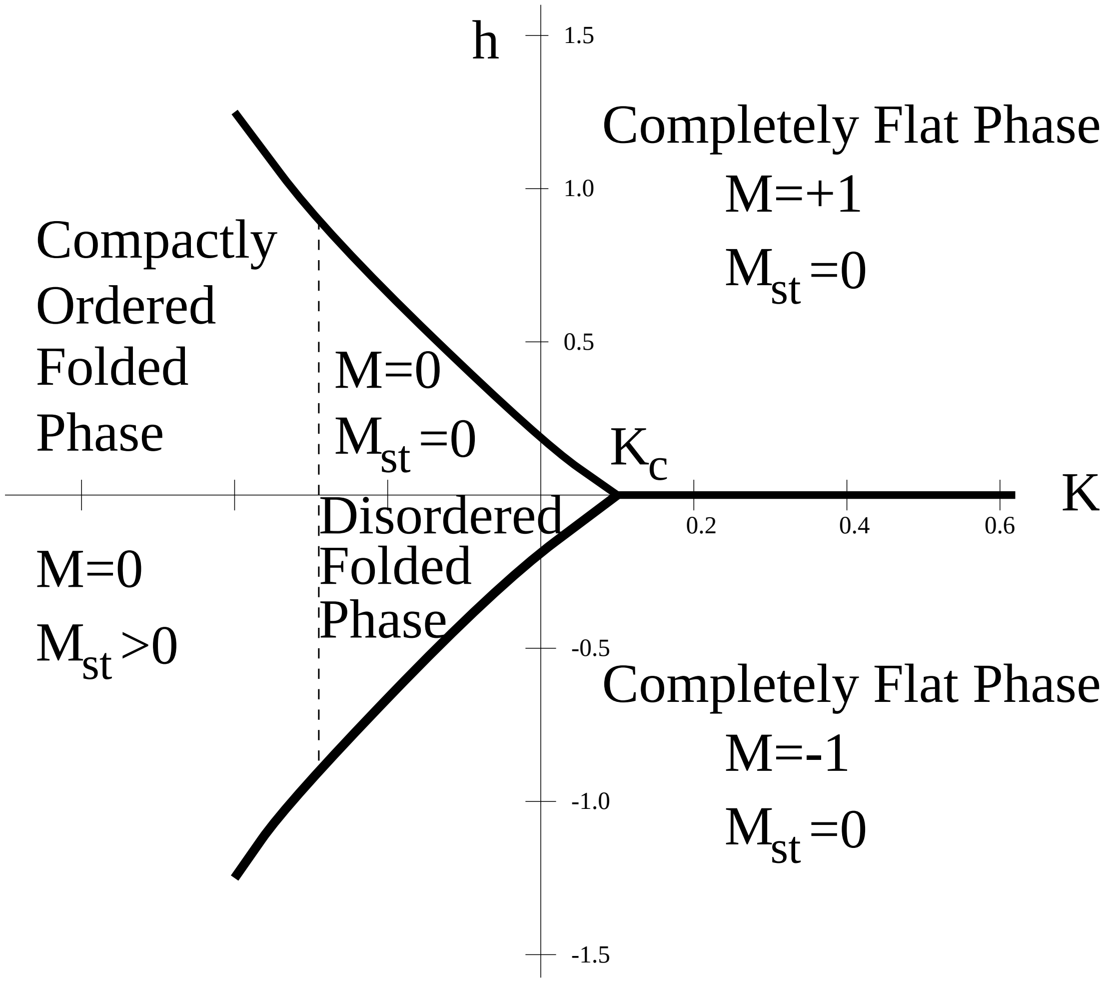
(2) Possible for Piled Configuration: Reversing All Spins in an Hexagon

Entropy or Number of States N_S for N Triangles:

$$N_S \sim q^N > 2^{\frac{1}{6} \times N} \tag{22}$$

$$q = \frac{\sqrt{3}}{2\pi} \Gamma(1/3)^{\frac{3}{2}} = 1.208717 \cdots \tag{23}$$

Phase Diagram in the (K, h)-plane:(Ph. Di Francesco, E.Guitter and S.Mori etc)



$$M = \frac{1}{N} < (\sum_{\triangle} S_i + \sum_{\triangledown} S_i) > \qquad (24)$$

$$M_{st} = \frac{1}{N} < (\sum_{\triangle} S_i - \sum_{\triangledown} S_i) > \qquad (25)$$

Three-Coloring Formulation:

The Folding Problem is equivalent to 3-coloring Problem of the Bonds

Three Colors R (0), G (1),B (2), ordered cyclically (modulo 3).

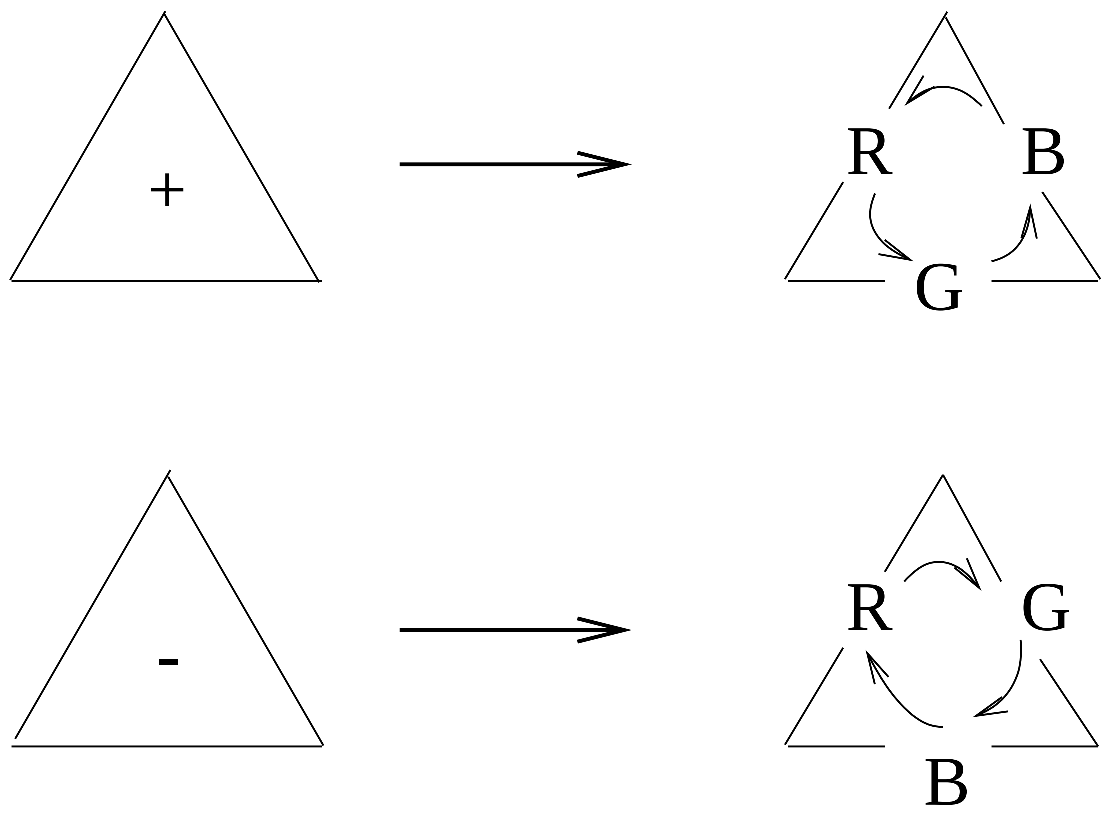


図 10: Color Assignments for Bonds

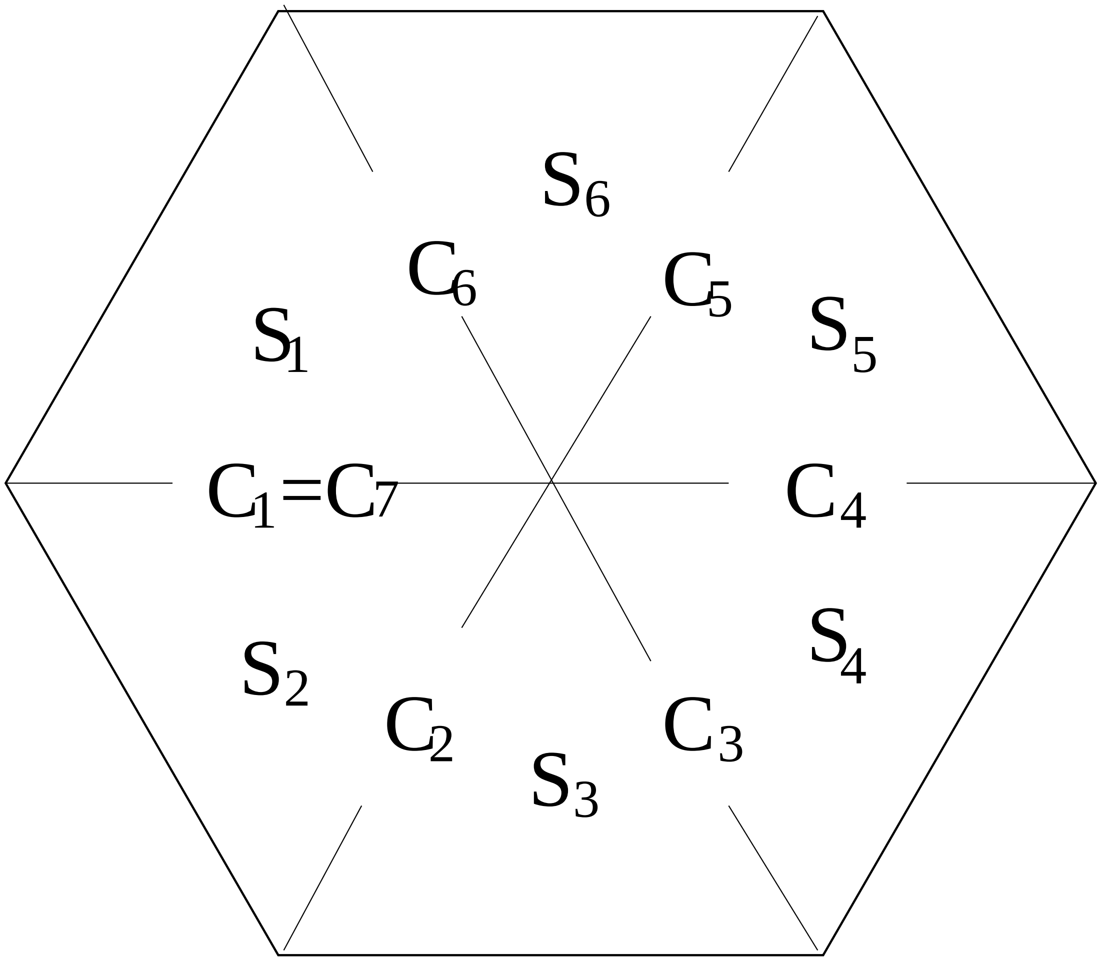


図 11: Coloring is Consistent if $c_7 = c_1$

$$C_{i+1} = C_i + S_i \mod 3$$

After One Turn,

$$C_7 = C_1 \quad \sum_{i=1}^{6} S_i = 0 \mod 3$$

4 Folding of the Foldable Randomly Triangulated Surface

Fluid Membrane= Randomly Triangulated Surface + Foldability

Foldable Triangulated Surface = Eulerian Triangulated Surface (Planar Surface Case)

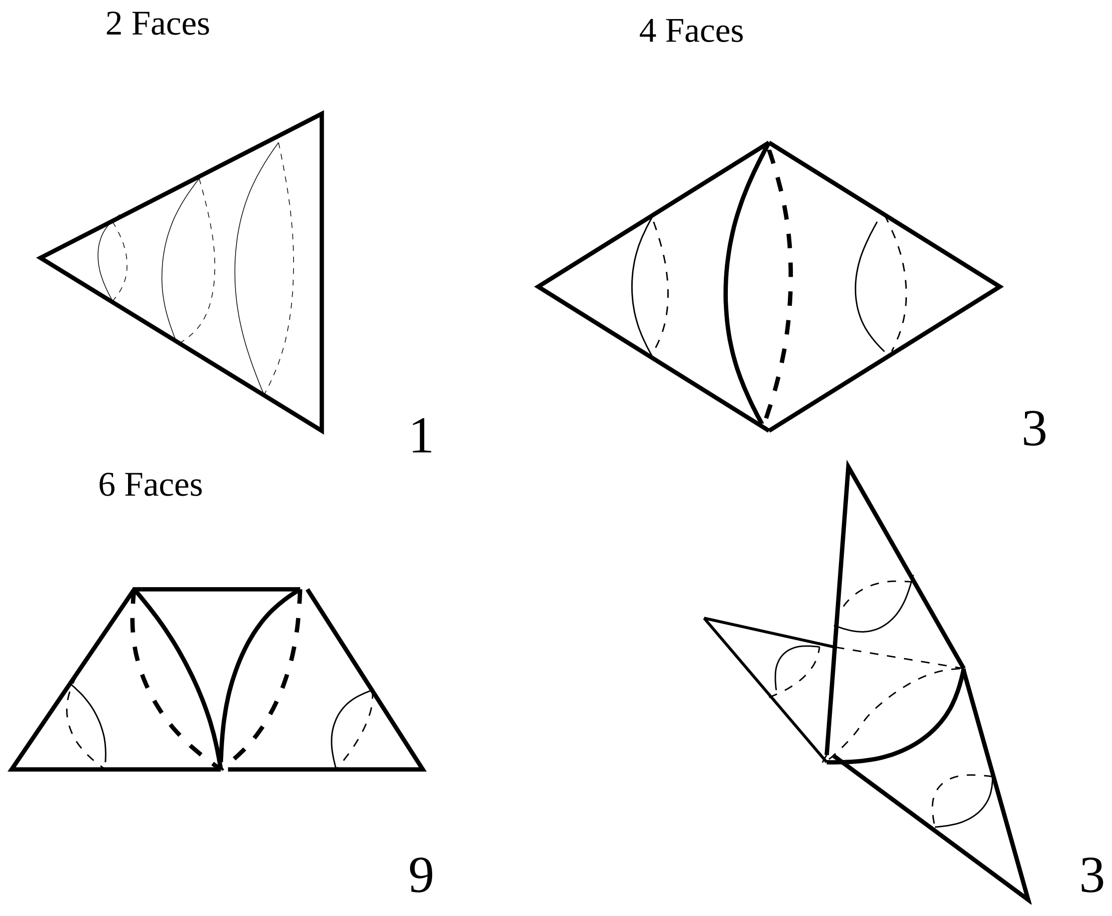


図 12: Eulerian Triangulated Sphere

4.1 Matrix Model

Planar Eulerian Triangulated Surface = Dual Graph of Bipartite Tri-valent Planar Graphs

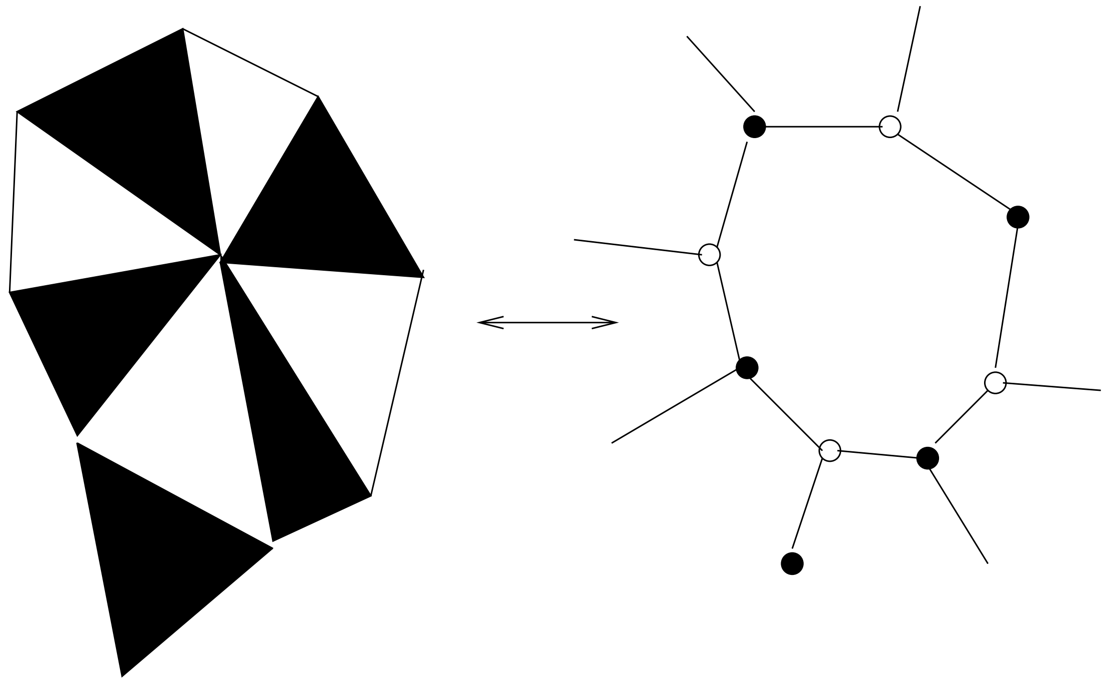


図 13: Eulerian Trianulated Surface and its Dual

Generating Function in Matrix Model :

$$Z_N(t) = \int dAdBe^{-N\mathrm{Tr}(V(A,B))}$$
$$V(A,B) = AB - \frac{\sqrt{t}}{3}(A^3 + B^3) \tag{26}$$

A,B: Hermitian Matrices of size $N \times N$

The Generating Function for Rooted Objects is

$$E(t) = kt\frac{d}{dt}F_0(t)$$

$$F_0(t) = \lim_{N\to\infty}\frac{1}{N^2}\log Z_N(t) \tag{27}$$

Results by W.Tutte "A Census of Planar Maps " (1963)

$$E(t) = \frac{1}{32t^2}((1-8t)^{\frac{3}{2}} - 1 + 12t + 24t^2) = \sum_{n=1} e_n t^n \tag{28}$$

and

$$e_n = \frac{3\cdot 2^{n-1}}{(n+1)(n+2)}{}_{2n}C_n = \frac{3}{n+2}\times 2^{n-1}C_n \tag{29}$$

$$e_1 = 1, \quad e_2 = 3, \quad e_3 = 12 \tag{30}$$

4.2 Toy Models

Toy Model for Polymer Membrane:

- Folding of Triangular Lattice
- Constrained Z_2 Spin System on the Dual of Triangular Lattice
- 3-Coloring Problem of Triangular Lattice

Toy Model for Fluid Membrane:

- Folding of Eulerian Randomly Triangulated Surface
- Constrained Z_2 Spin System on its Dual Random Diagram
- 3-Coloring Problem of the Random Diagram

How to Formulate.

- Matrix Model ?
  Difficult and not yet solved
- Decorated Tree
  also difficult and under progress

4.3 Combinatorial Solution by Decorated Tree

One-to-One Correspondence between Eulerian Triangulation and Well-Balanced Tree

by M.Bousquet-Mélow and G.Schaeffer (2000), J.Bouttier, P.Di Francesco and E.Guitter (2002/cond-mat/0206452)

Steps:

- Rooted Binary Tree with n White vertices, $n-1$ edges and $n+2$ white leaves (including the root)

$$C_n = \frac{1}{n+2}{}_{2n}C_n \tag{31}$$

- Decorated by $n-1$ Black verteices and Black Buds to each edge

  $2^{n-1}C_n$

- Match Buds to the closest Leaves in counter clockwise direction (Planar)

  $n-1$ Buds and Leaves Pair and 3 unmathced White leaves

- Well-balanced if the Root remains unmathced

$$\frac{3}{n+2}2^{n-1}C_n = e_n$$

From Decorated Tree to Rooted Bipartite Trivalent Planar Graph

- Fusing Bud-Leaf Pair into Edge
- Three unmatched leaves are connected to Additional Black Vertex
- Marked edge is that pointing from this vertex to the root leaf

Rooted Binary Tree

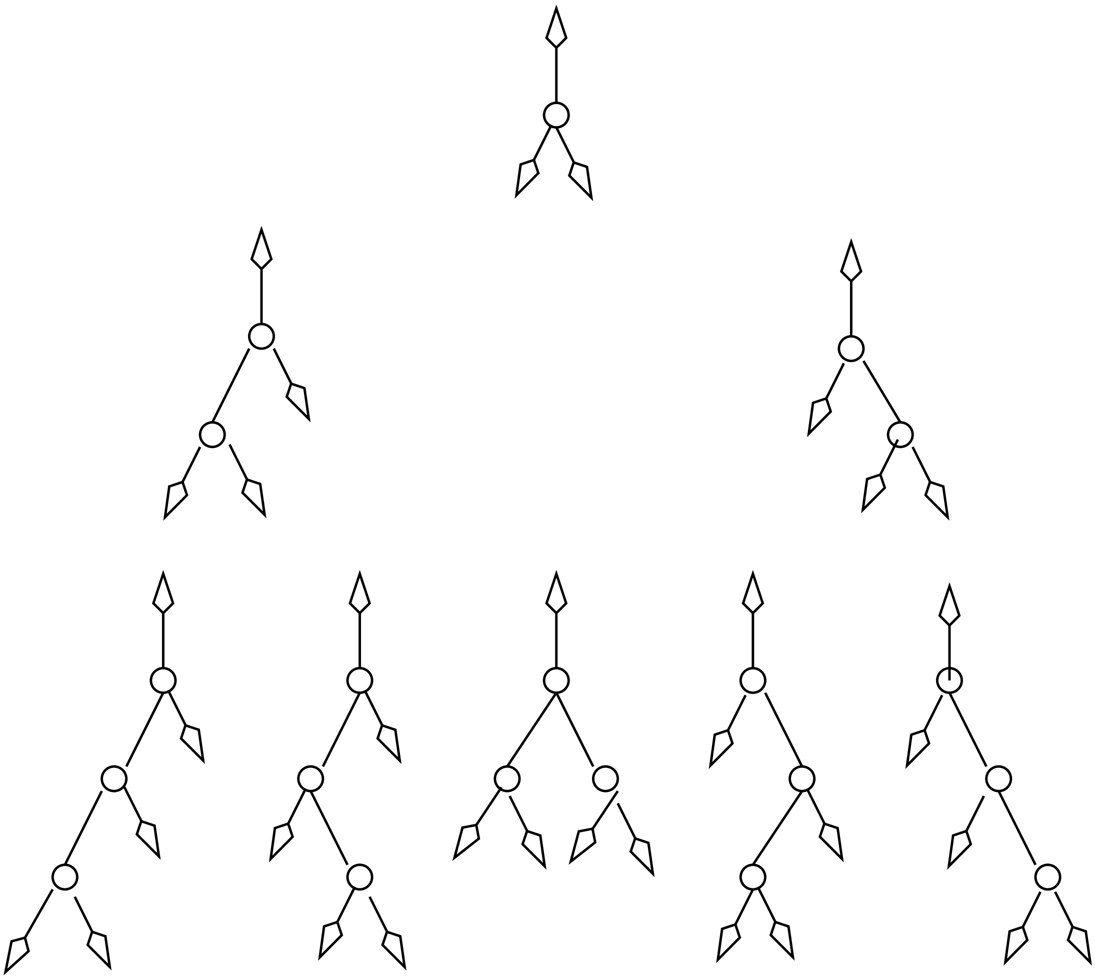


図 14: Binary Tree

Recursion Relation:

$$C_n = \sum_{i=n-1}^{0} C_i \times C_{n-1-i}$$

Decoration

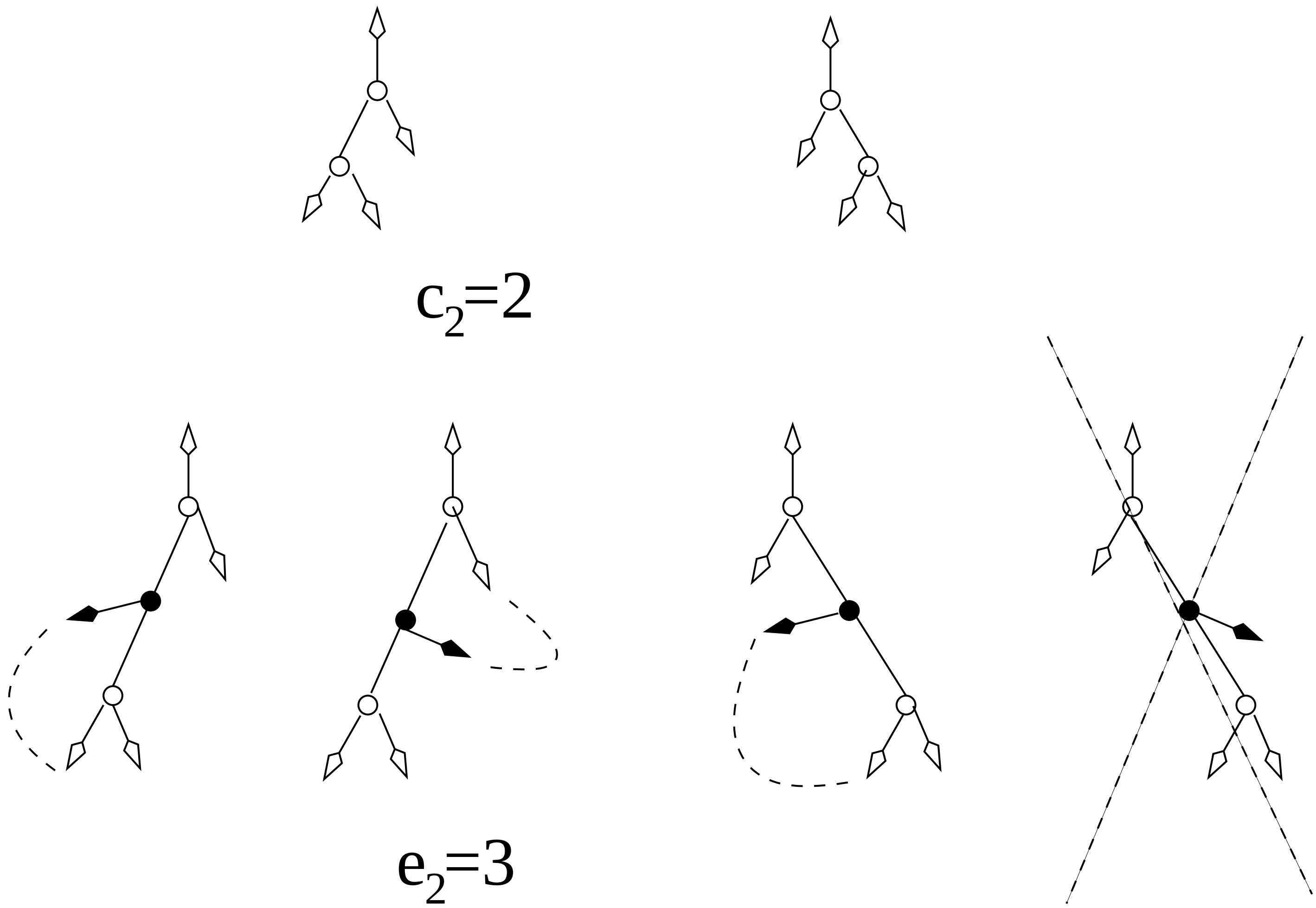


図 15: Decorated Binary Tree

Decorated Tree to Trivalent Graph

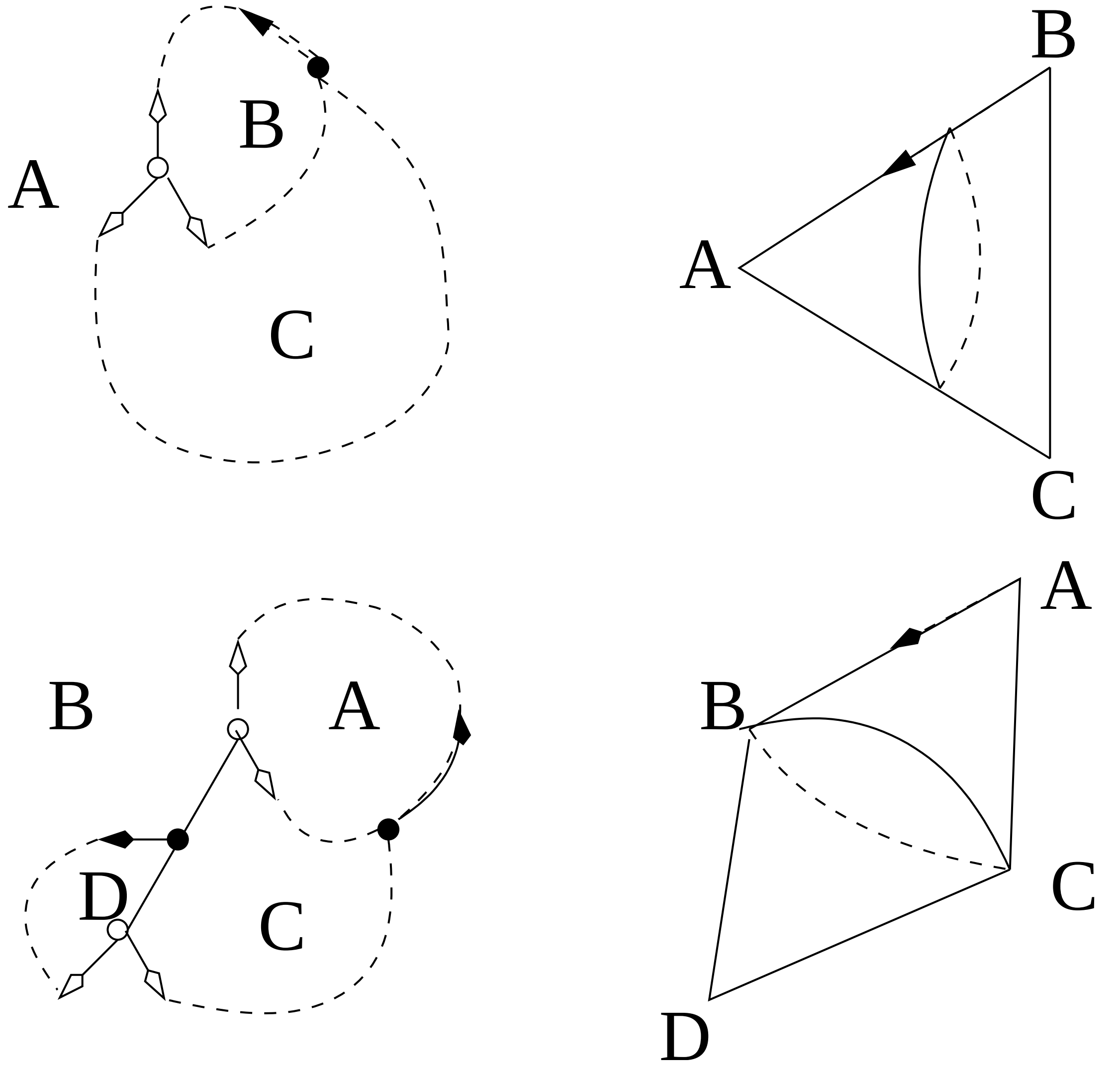


図 16: Decorated Binary Tree to Trivalent Tree and Triangulated Surface

Decorated Tree to Trivalent Graph

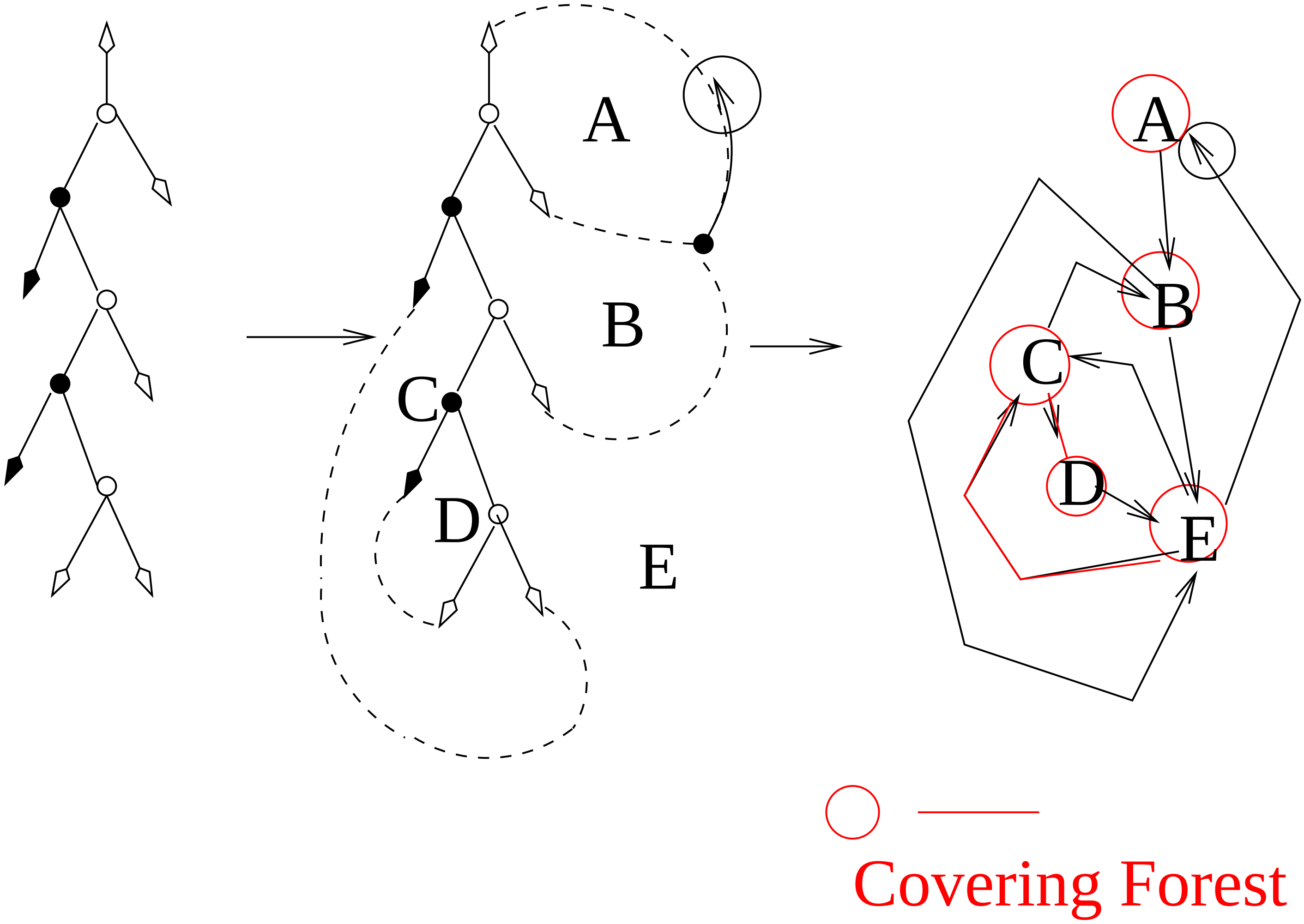


図 17: Decorated Binary Tree to Trivalent Tree and Triangulated Surface and Covering Forest

Under Construction(秋の物理学会の高分子分科のほうで)

5 Future Problem

- How to Describe Folding Path of Polymer
  Not “Census “, but “Number of Folding Path”
  “Physical” Move from Meander to Meander

- Physical Properties of Randomly Triangulated Surface
  (1) Entropy
  (2) Bending Energy and Phase Diagram
  (3) How to Implement on Computer