# ランダムに三角形分割された膜の統計物理

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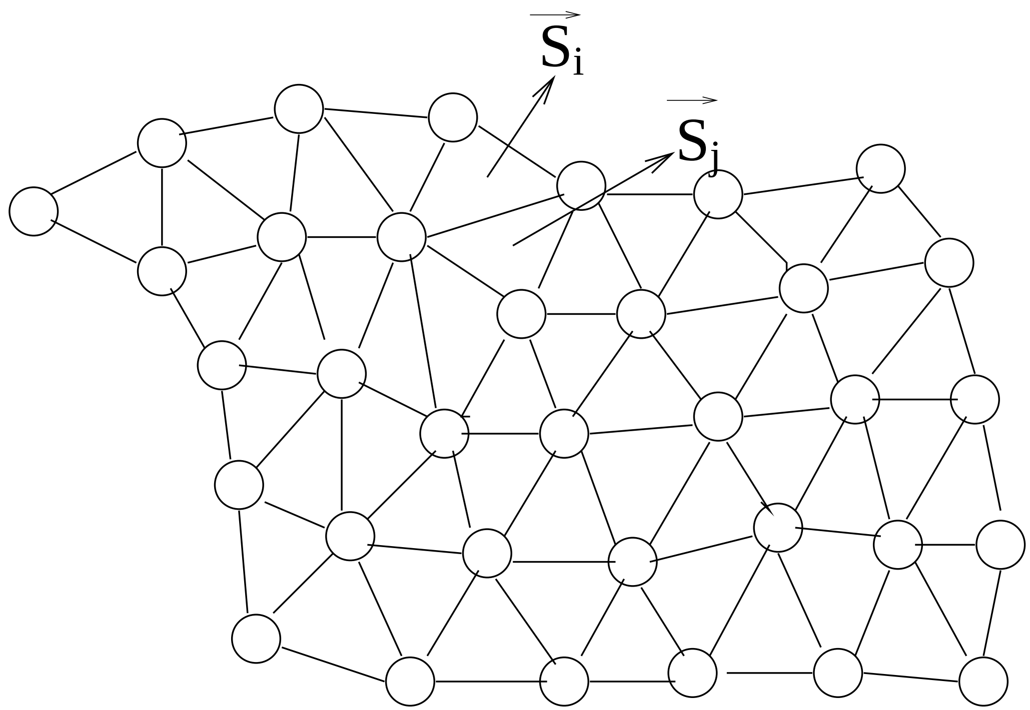
- Fluid Membrane embedded in 2-D Space
- Model : Folding of Randomly Triangulated Surface
- Analysis : Cluster Variation Method
- Results (Entropy) and Future Problem (Phase Diagram etc.)

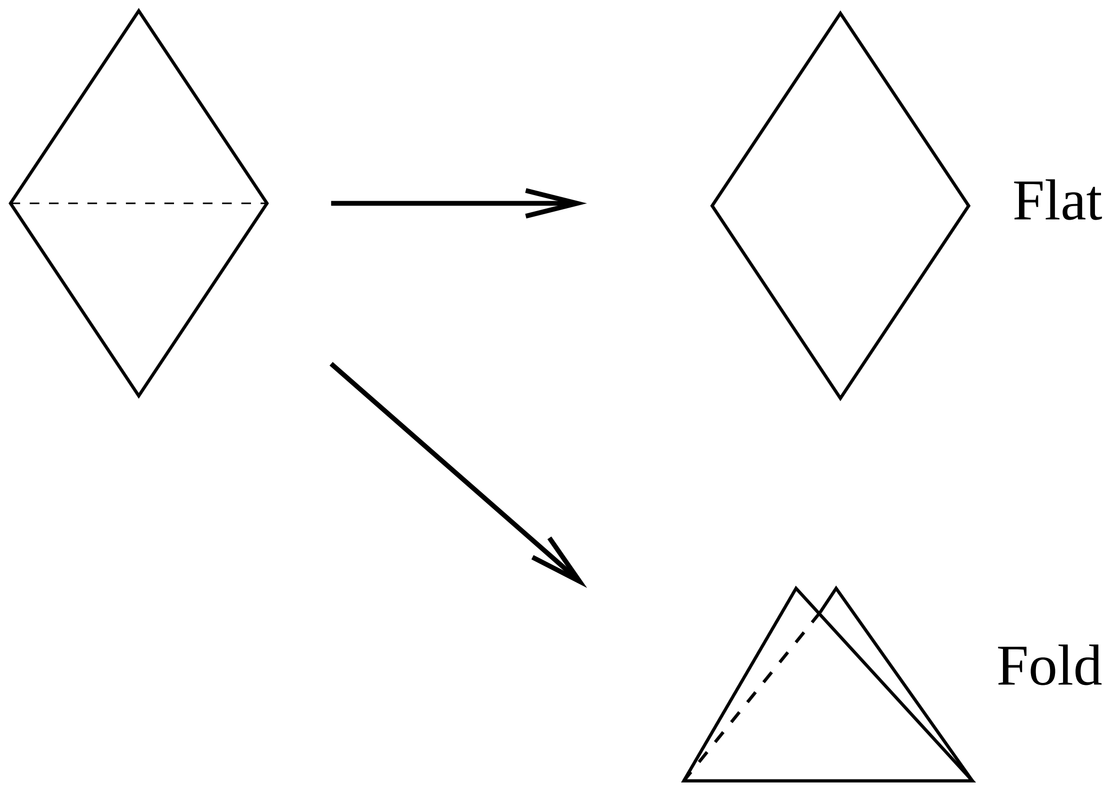
# 1 Fluid Membrane embedded in 2-D Space

## 1.1 Membrane ?



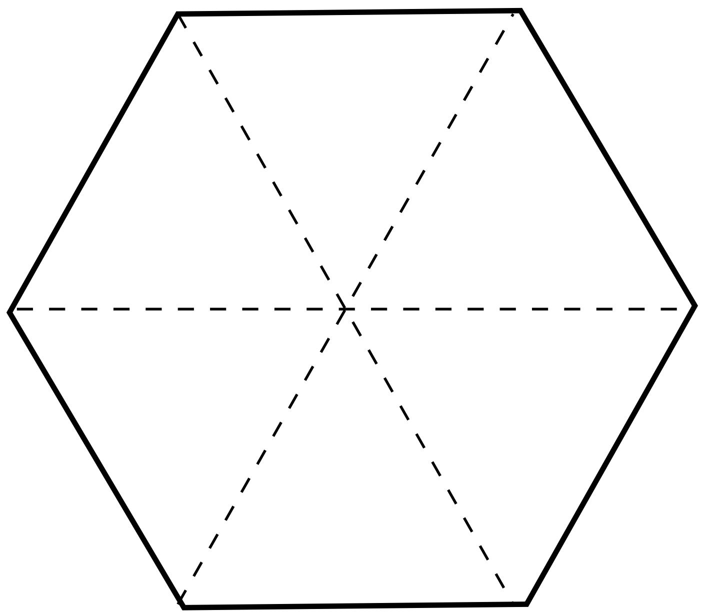

$$E = -J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \tag{1}$$

Embedded in 2-dimensional Space → Fold or Flat




Even Phantom Case is not Simple !

Folding of Elementary Hexagon:



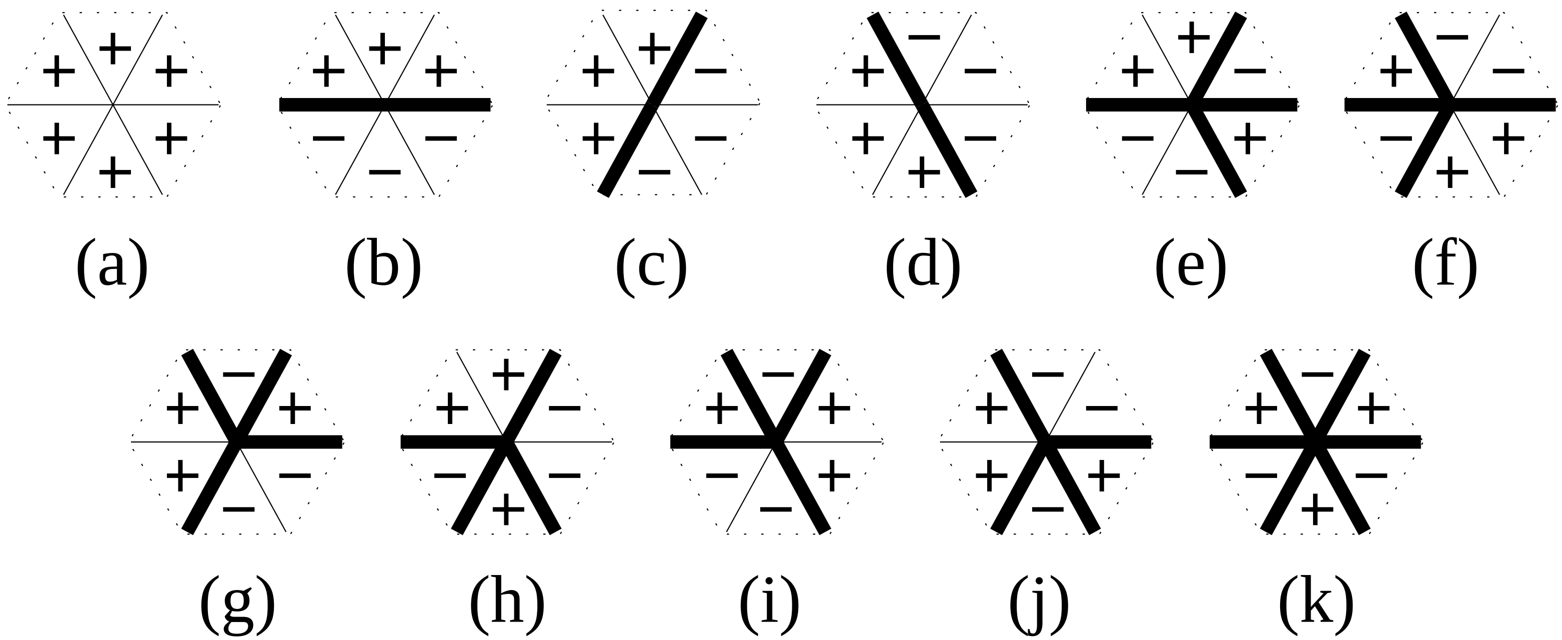
6 Bonds ----> 64 States ?

Only 11 Physical States

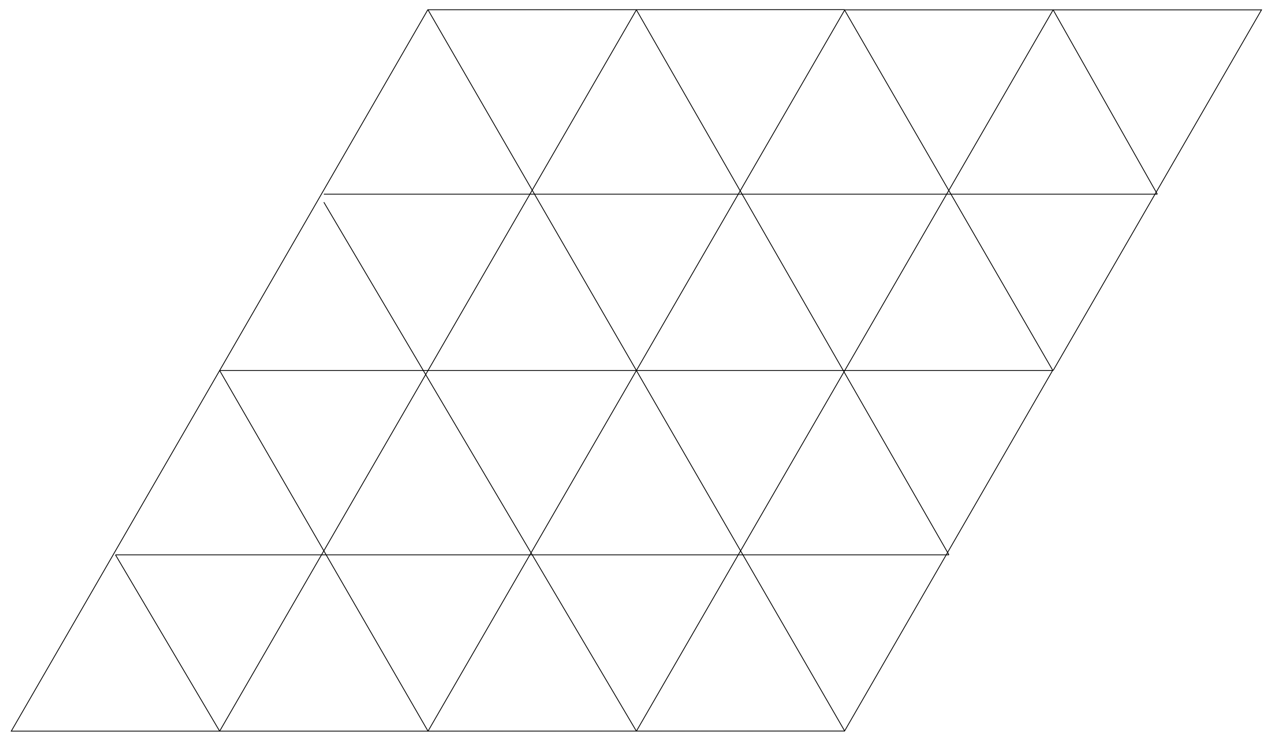
11 local fold environments for elementary Hexagon

Face Up $- - - - - - - - - - - - - - - > S = +1$

Face Down $- - - - - - - - - - - - - - -> S = -1$



# Regular Triangular Lattice



Embedded in 2-dimensional Space → Fold or Flat

Model System:

Folding Constraints :

$$\sigma_v \equiv \sum_{i \ \ around \ \ v} S_i = 0 \mod 3 \tag{2}$$

Constrained $Z_2$ Spin System : 11 vertex model

Hamiltonian:

$$\mathcal{H}_{Ising} = -J \sum_{<ij>} S_i S_j - H \sum_i S_i \tag{3}$$

$$K = \beta J = J/k_B T \quad \text{and} \quad h = \beta H \tag{4}$$

Entropy or Number of States $N_S$ for $N$ Triangles:

$$N_S \sim q^N > 2^{\frac{1}{6} \times N} \tag{5}$$

$$q = \frac{\sqrt{3}}{2\pi}\Gamma(1/3)^{\frac{3}{2}} = 1.208717\cdots \tag{6}$$

Ph. Di Francesco and E.Guitter

Fluid Membrane = Randomly Triangulated Surface + Foldability

Foldable Triangulated Surface = Eulerian Triangulated Surface (Planar Surface Case )

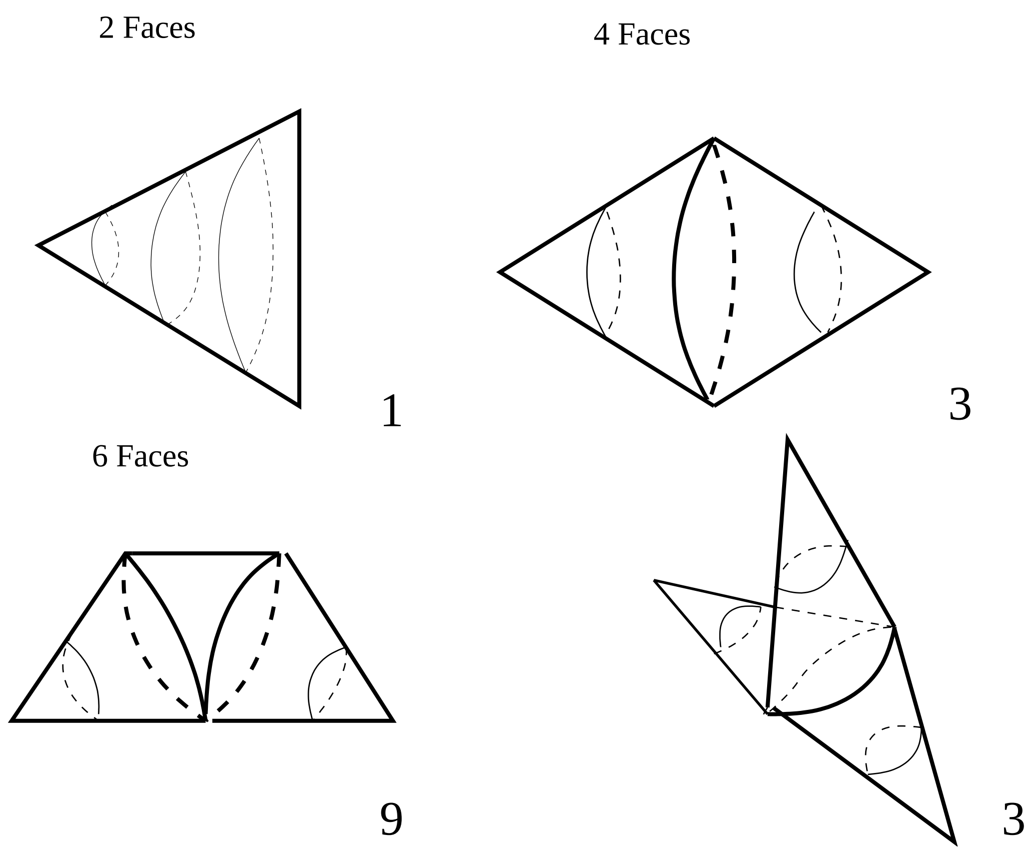
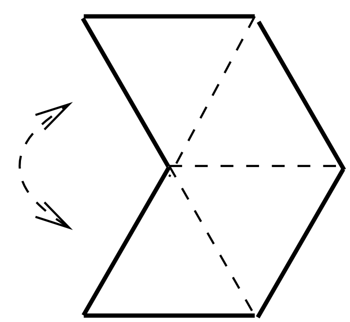


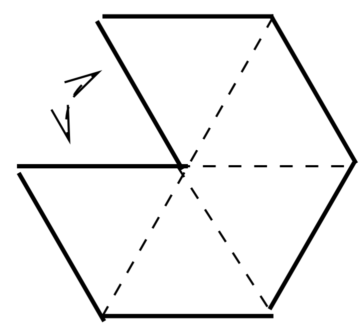

図 1: Eulerian Triangulated Sphere

Is It Foldable ?

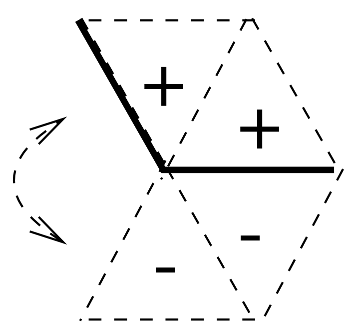
Foldable



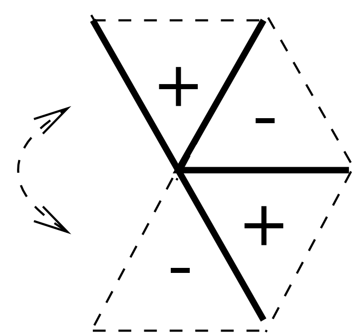
Not Fordable



4 Bonds   3 States



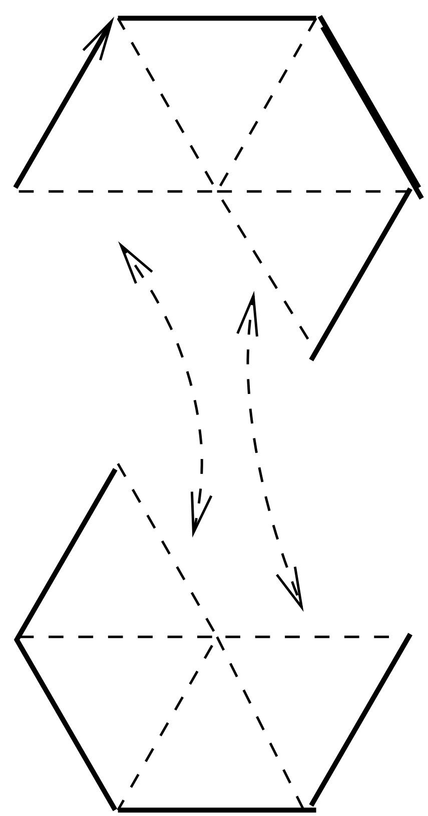

2

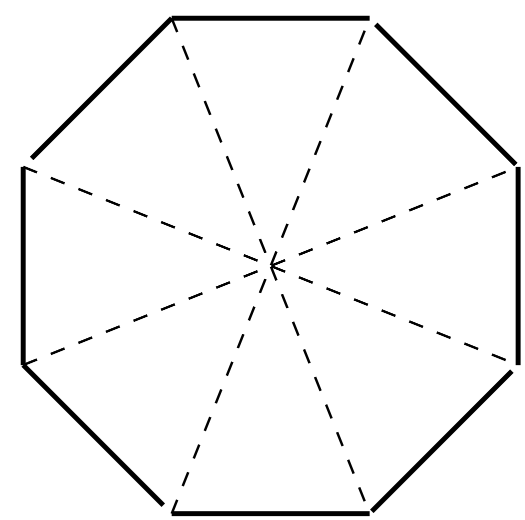



1

8-Triangles Case:

Foldable





8 Bonds

43 States

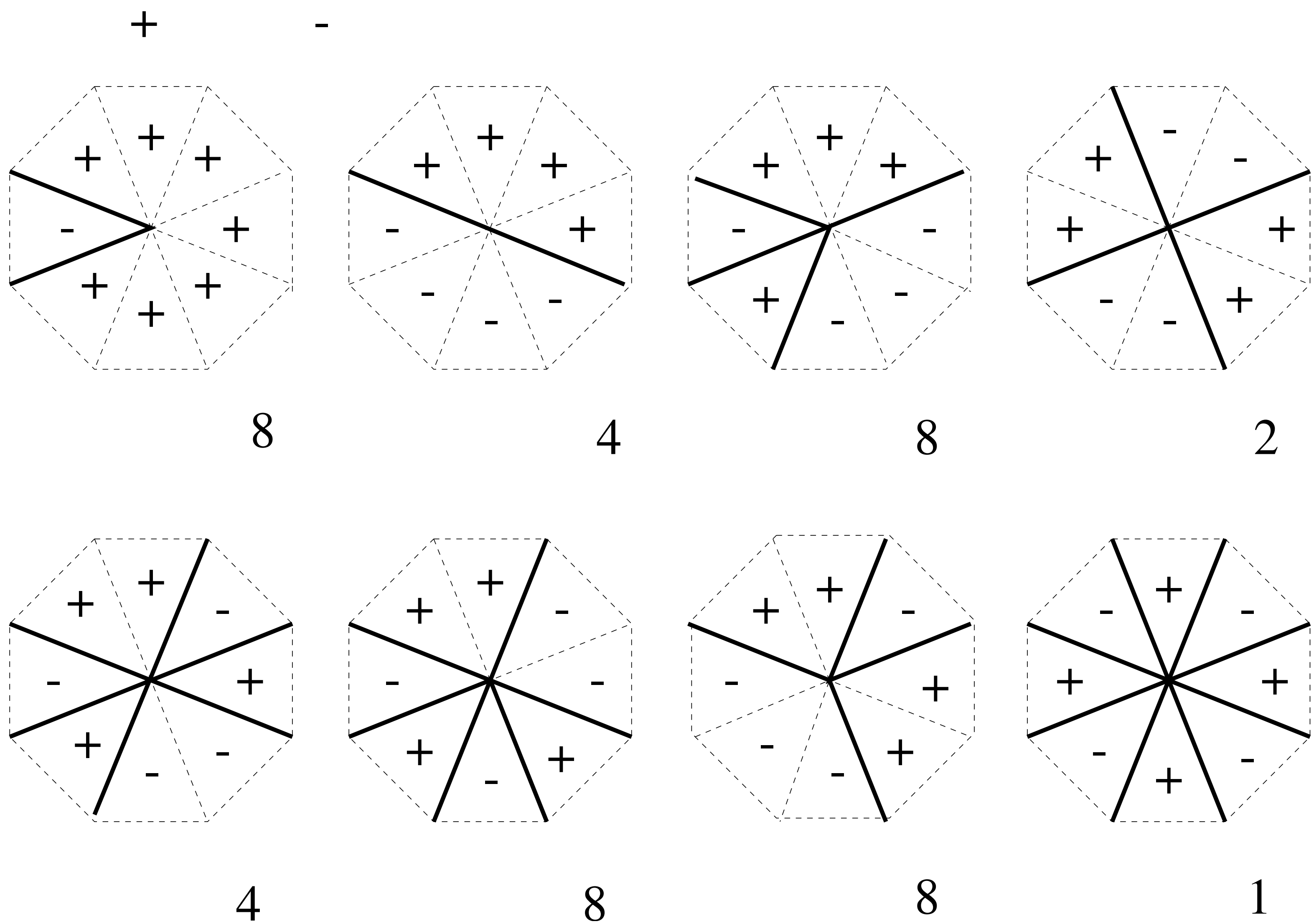



図 2: 43 patterns for the Vertex composed of 8 Triangles.

# 2 Model : Folding of Randomly Triangulated Surface

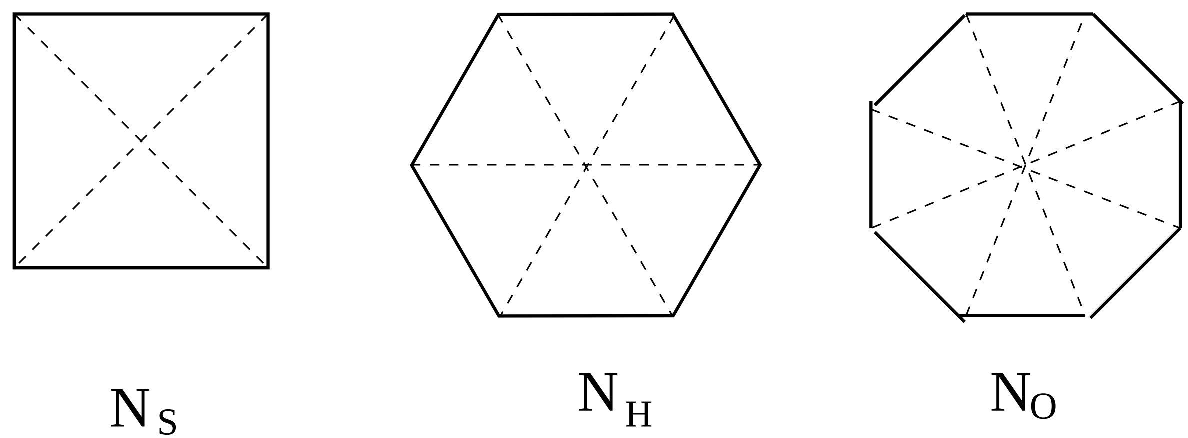



図 3: Distribution of Square, Hexagon and Octagon

$$\rho_4(S_1,S_2,S_3,S_4) \quad \rho_6(S_1,S_2,S_3,S_4,S_5,S_6) \quad \rho_8(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8) \tag{7}$$

$$\sum_{i=0}^{4,6,8} S_i = 0 \mod 3$$

$$\begin{aligned} N &= \text{Number of Vertex (Spin) or Number of Triangle} \\ N_E &= \frac{3}{2}N = \text{Number of Edges or Number Spin Pair} \\ N_F &= \text{Number of Faces} = N_S + N_H + N_O \end{aligned} \tag{8}$$

From Euler's Relation,

$$N - \frac{3}{2}N + N_F = 2 \quad \longrightarrow \quad N_F = \frac{1}{2}N$$

and the identity

$$\frac{1}{2}(4 \times N_S + 6 \times N_H + 8 \times N_H) = N_E$$

Introducing $\alpha$ and $\beta = 1 - 2\alpha$,

$$N_S = N_O = \frac{1}{2}N \times \alpha \;\text{ and }\; N_H = \frac{1}{2}N \times \beta$$

# 3 Analysis : Cluster Variation Method

$$
\begin{aligned}
f(\rho_4,\rho_6,\rho_8) = &-\frac{3}{2}K\mathrm{Tr}_{1,2}\rho_2(S_1,S_2)S_1S_2 - \frac{1}{2}h\mathrm{Tr}_1\rho_{1A}(S_1)S_1 - \frac{1}{2}h\mathrm{Tr}_2\rho_{1B}(S_2)S_2 \\
&+ \frac{1}{2}\mathrm{Tr}_1\rho_{1A}(S_1)\log\rho_{1A}(S_1) + \frac{1}{2}\mathrm{Tr}_2\rho_{1B}(S_2)\log\rho_{1B}(S_2) \\
&- \frac{3}{2}\mathrm{Tr}_{1,2}\rho_2(S_1,S_2)\log\rho_2(S_1,S_2) \\
&+ \frac{\alpha}{2}\mathrm{Tr}_{1,2,3,4}\rho_4\log\rho_4 + \frac{\beta}{2}\mathrm{Tr}_{1,2,3,4,5,6}\rho_6\log\rho_6 \\
&+ \frac{\alpha}{2}\mathrm{Tr}_{1,2,3,4,5,6,7,8}\rho_8\log\rho_8 + \lambda_8(\mathrm{Tr}_{1,2,3,4,5,6,7,8}\rho_8 - 1) \\
&+ \lambda_6(\mathrm{Tr}_{1,2,3,4,5,6}\rho_6 - 1) + \lambda_4(\mathrm{Tr}_{1,2,3,4}\rho_4 - 1)
\end{aligned} \tag{9}
$$

$$
P_2(S_1,S_2) = \frac{2}{3}\alpha\mathrm{STr}^{''}P_4 + \beta\mathrm{STr}^{''}P_6 + \frac{4}{3}\alpha\mathrm{STr}^{''}P_8 \tag{10}
$$

$$
P_{1A}(S_1) = \mathrm{Tr}_2P_2(S_1,S_2) \quad \text{and} \quad P_{1B}(S_1) = \mathrm{Tr}_1P_2(S_1,S_2) \tag{11}
$$

Reduction from $P_4, P_6, P_8$ to $P_2$

$$\mathrm{STr}^{''} P_4 = \frac{1}{4} \sum_{3,4} (P_4(S_1, S_2, S_3, S_4) + P_4(S_3, S_2, S_1, S_4) + P_4(S_3, S_4, S_1, S_2) + P_4(S_1, S_3, S_4, S_2)) \tag{12}$$

$$\mathrm{STr}^{''} P_6 = \frac{1}{6} \sum_{3,4,5,6} (P_6(S_1, S_2, S_3, S_4, S_5, S_6) + P_6(S_3, S_2, S_1, S_4, S_5, S_6) + P_6(S_3, S_4, S_1, S_2, S_5, S_6) + P_6(S_3, S_4, S_5, S_2, S_1, S_6) + P_6(S_3, S_4, S_5, S_6, S_1, S_2) + P_6(S_1, S_3, S_4, S_5, S_6, S_2)) \tag{13}$$

$$\mathrm{STr}^{''} P_8 = \frac{1}{8} \sum_{3,4,5,6,7,8} (P_8(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) + P_8(S_3, S_2, S_1, S_4, S_5, S_6, S_7, S_8) + P_8(S_3, S_4, S_1, S_2, S_5, S_6, S_7, S_8) + P_8(S_3, S_4, S_5, S_2, S_1, S_6, S_7, S_8) + P_8(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) + P_8(S_3, S_4, S_5, S_6, S_1, S_2, S_7, S_8) + P_8(S_3, S_4, S_5, S_6, S_7, S_2, S_1, S_8) + P_8(S_1, S_3, S_4, S_5, S_6, S_7, S_8, S_2)) \tag{14}$$

Natural Iteration Method:

$$
\begin{aligned}
\rho_6 \quad & (S_1, S_2, S_3, S_4, S_5, S_6) = \exp(-\lambda_6 + \frac{K}{2}\sum_{i=1}^{6} S_i S_{i+1} + \frac{h}{3}\sum_{i=1}^{6} S_i) \\
& \times (\rho_2(S_1, S_6)\rho_2(S_1, S_2)\rho_2(S_3, S_2)\rho_2(S_3, S_4)\rho_2(S_5, S_6)\rho_2(S_5, S_4))^{\frac{1}{2}} \\
& \times (\rho_{1A}(S_1)\rho_{1B}(S_2)\rho_{1A}(S_3)\rho_{1B}(S_4)\rho_{1A}(S_5)\rho_{1B}(S_6))^{-\frac{1}{3}} \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\rho_8 \quad & (S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) = \exp(-\lambda_8 + \frac{K}{2}\sum_{i=1}^{8} S_i S_{i+1} + \frac{h}{3}\sum_{i=1}^{8} S_i) \\
& \times (\rho_2(S_1, S_8)\rho_2(S_1, S_2)\rho_2(S_3, S_2)\rho_2(S_3, S_4) \\
& \rho_2(S_5, S_6)\rho_2(S_5, S_4)\rho_2(S_7, S_6)\rho_2(S_7, S_8))^{\frac{1}{2}} \\
& \times (\rho_{1A}(S_1)\rho_{1B}(S_2)\rho_{1A}(S_3)\rho_{1B}(S_4)\rho_{1A}(S_5)\rho_{1B}(S_6)\rho_{1A}(S_7)\rho_{1B}(S_8))^{-\frac{1}{3}} \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\rho_4 \quad & (S_1, S_2, S_3, S_4) = \exp(-\lambda_4 + \frac{K}{2}\sum_{i=1}^{4} S_i S_{i+1} + \frac{h}{3}\sum_{i=1}^{4} S_i) \\
& \times (\rho_2(S_1, S_4)\rho_2(S_1, S_2)\rho_2(S_3, S_2)\rho_2(S_3, S_4))^{\frac{1}{2}} \\
& \times (\rho_{1A}(S_1)\rho_{1B}(S_2)\rho_{1A}(S_3)\rho_{1B}(S_4))^{-\frac{1}{3}} \qquad (17)
\end{aligned}
$$

Entropy per Triangle:

$$s_6 = \frac{3}{6}\lambda_6 - \frac{3}{2} < e > \quad s_4 = \frac{3}{4}\lambda_4 - \frac{3}{2} < e > \quad s_8 = \frac{3}{8}\lambda_8 - \frac{3}{2} < e > \tag{18}$$

$$< e >=< -\frac{K}{2} \sum_{i=1}^{4,6,8} S_i S_{i+1} - \frac{h}{3} \sum_{i=1}^{4,6,8} S_i > \tag{19}$$

Total Entropy:

$$S^{Total} = N \times s = \frac{1}{3}(4 \times N_S \times s_4 + 6 \times N_H \times s_6 + 8 \times N_O \times s_8) \tag{20}$$

Averaged Entropy per Triangle:

$$s = (\frac{2}{3}\alpha s_4 + \beta s_6 + \frac{4}{3}\alpha s_8) \tag{21}$$

# 4 Results

## 4.1 Entropy

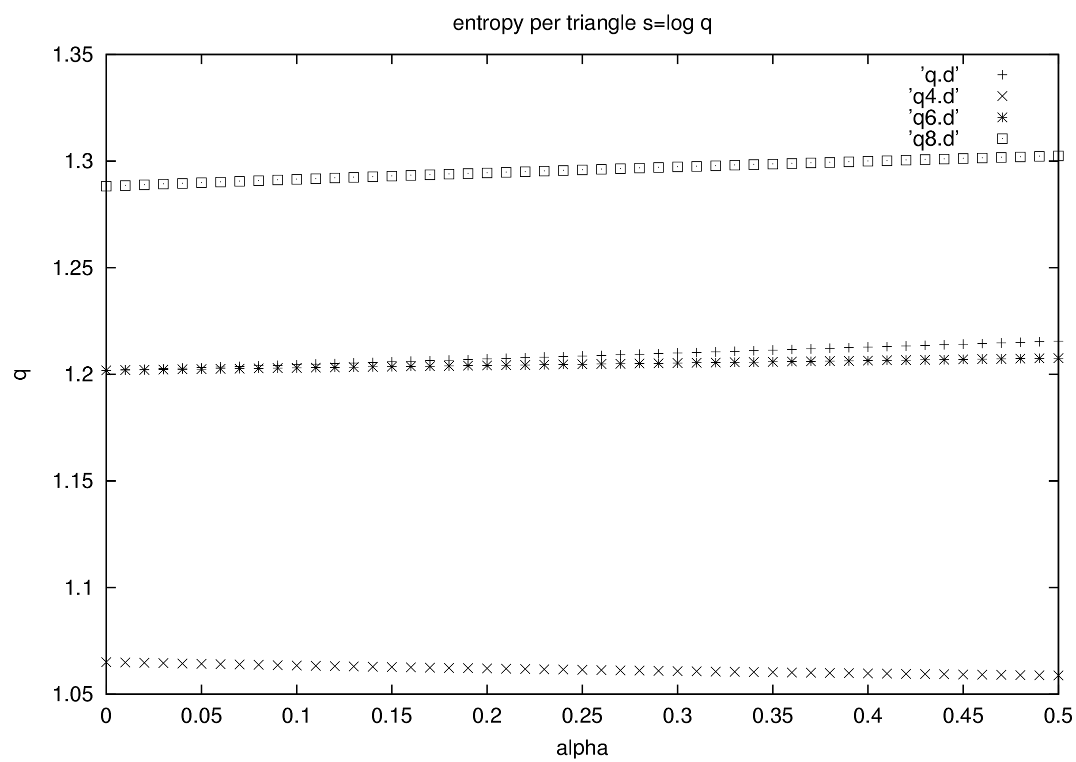



図 4: Entropy per Triangle for Square, Hexagon, Octagon

## 4.2 Nearest Neighbor Correlation $< S_i S_j >$

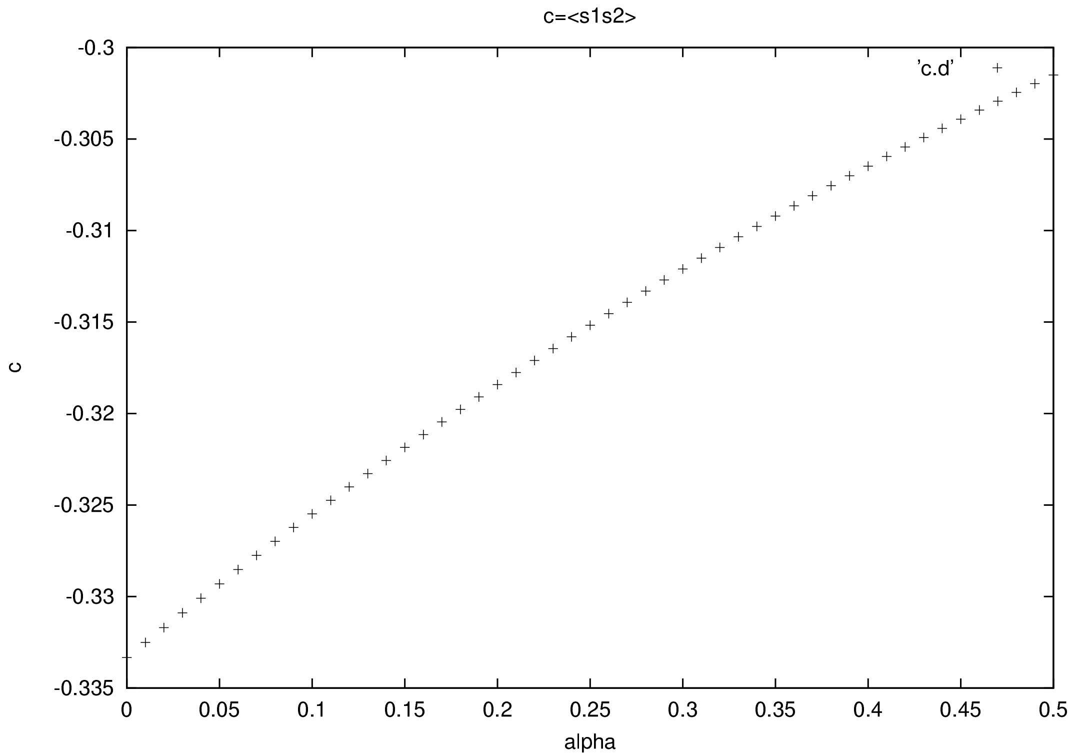
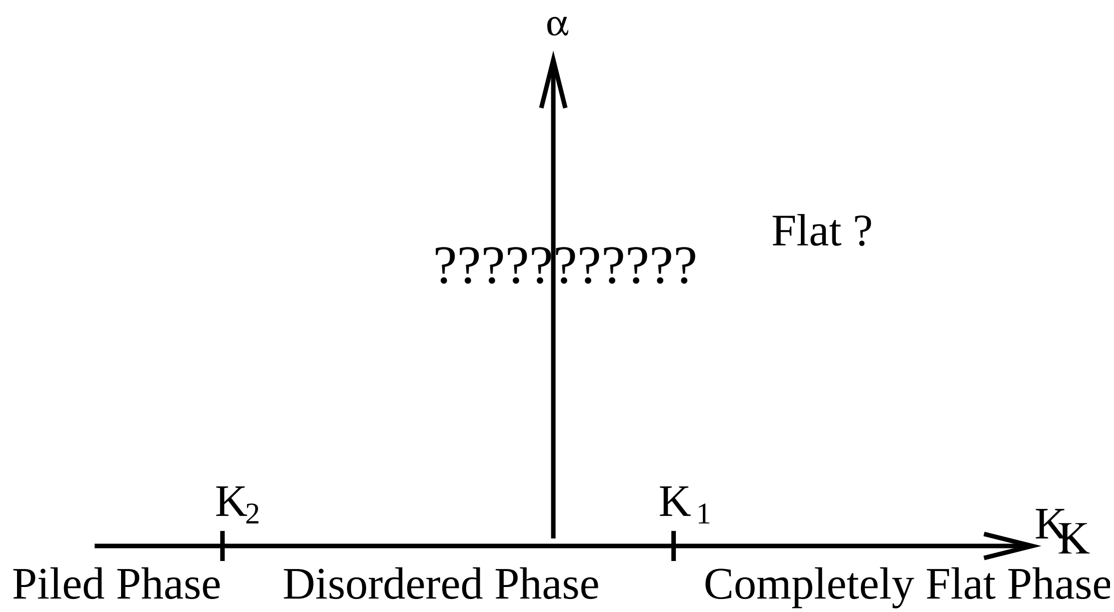



図 5: Nearet Neighbor Correlation Function

## 4.3 Future Problem

- Phase Diagram in $(K, \alpha)$ Plane




Frustration $--->$ Crumpling Transition exists ? First or Second Order ?

- Geometric Properties (Monte Carlo, Paralel-Tempering)
- Generation of (Real) Foldable Randomly Triangulated Surface