三角格子の折り畳み問題と液体膜への拡張

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龍谷大学 応用数理セミナー 特別研究集会

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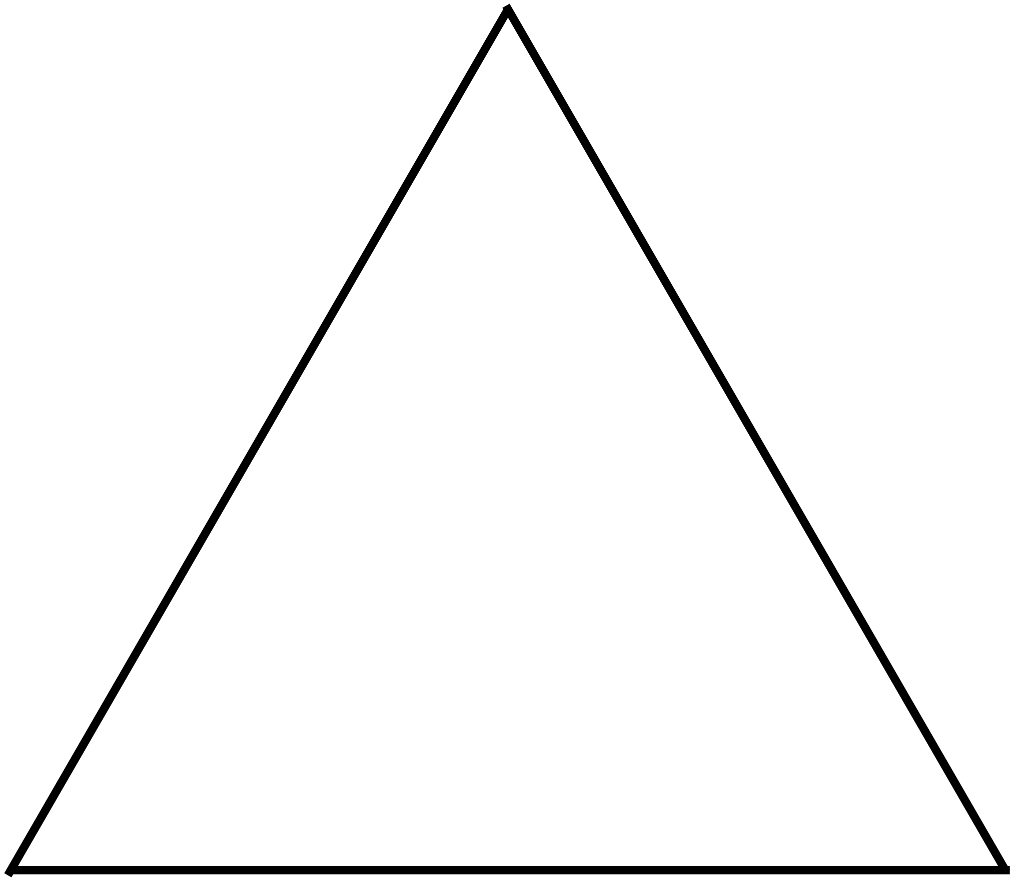
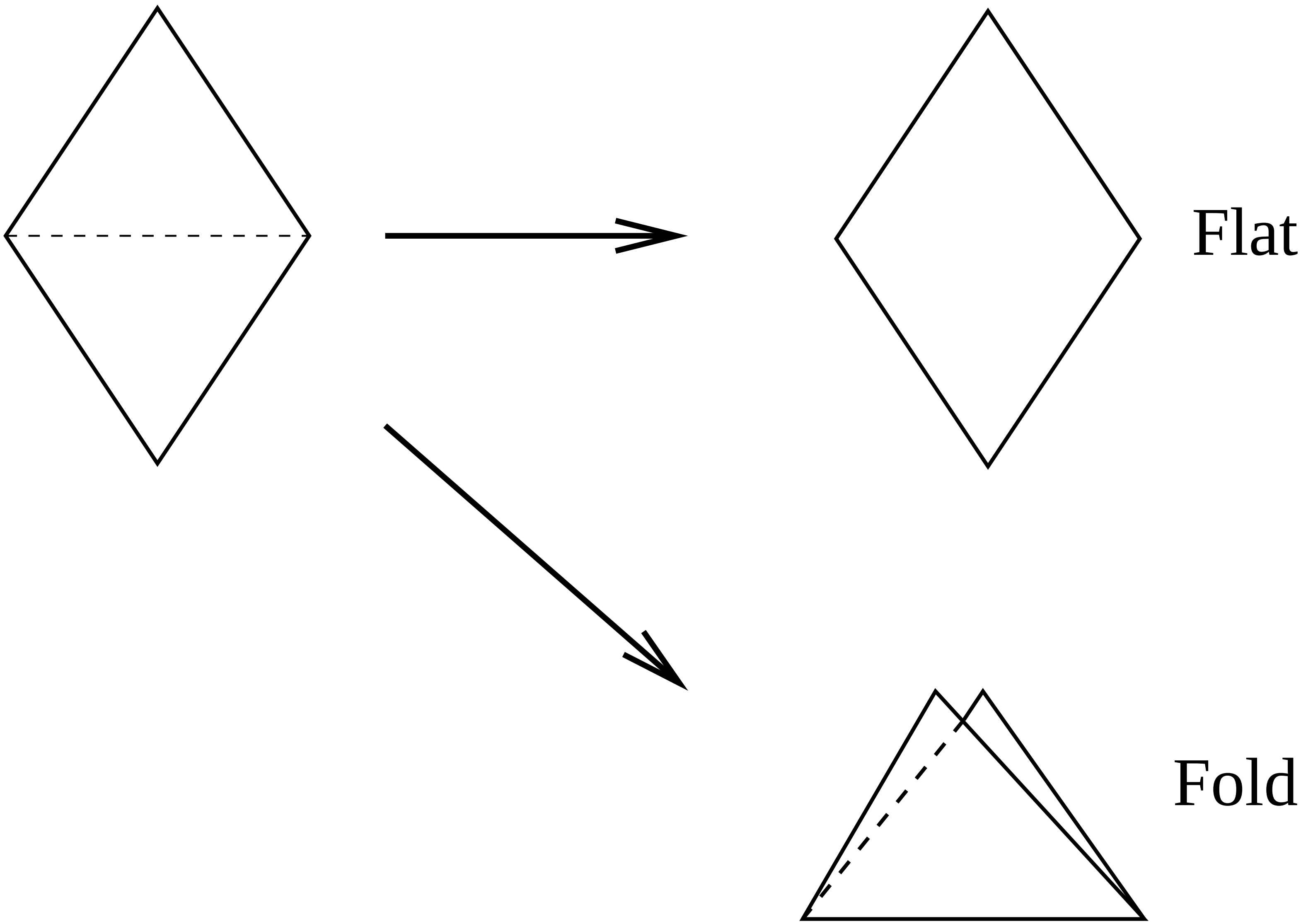
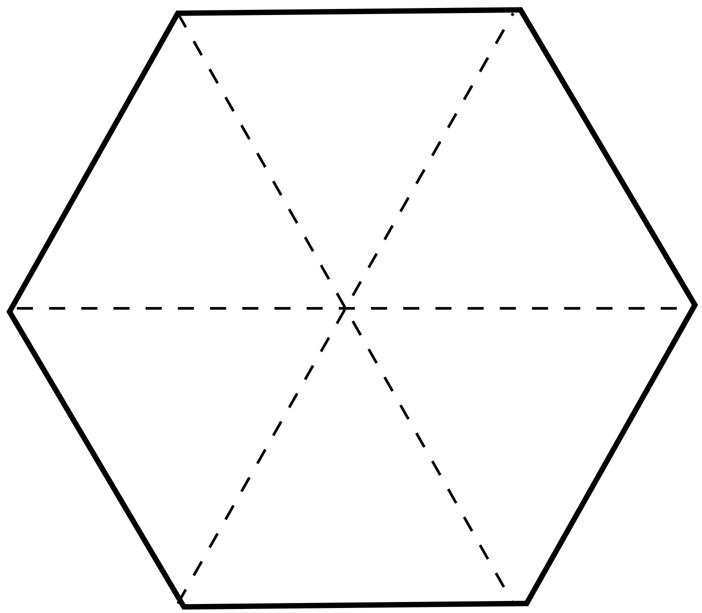
1　三角格子の折り畳みとは?



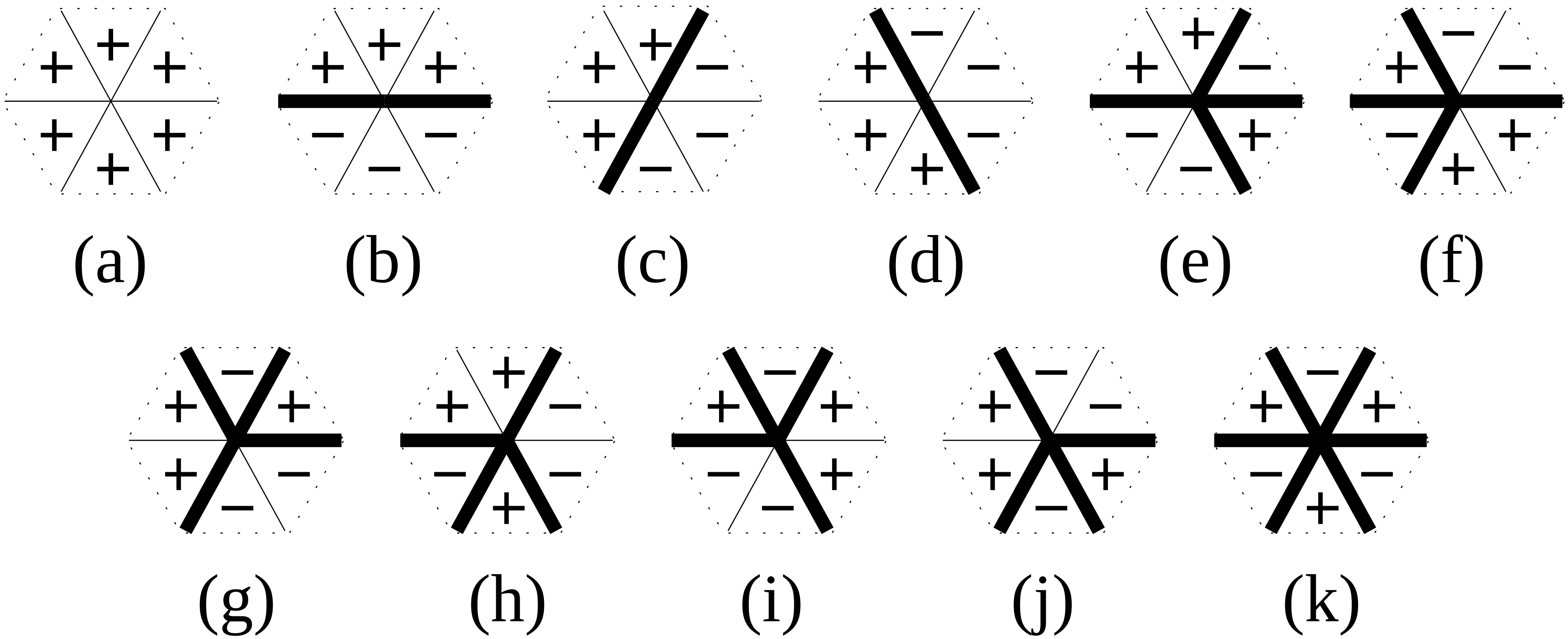

Flat
Fold

Elementary Hexagon



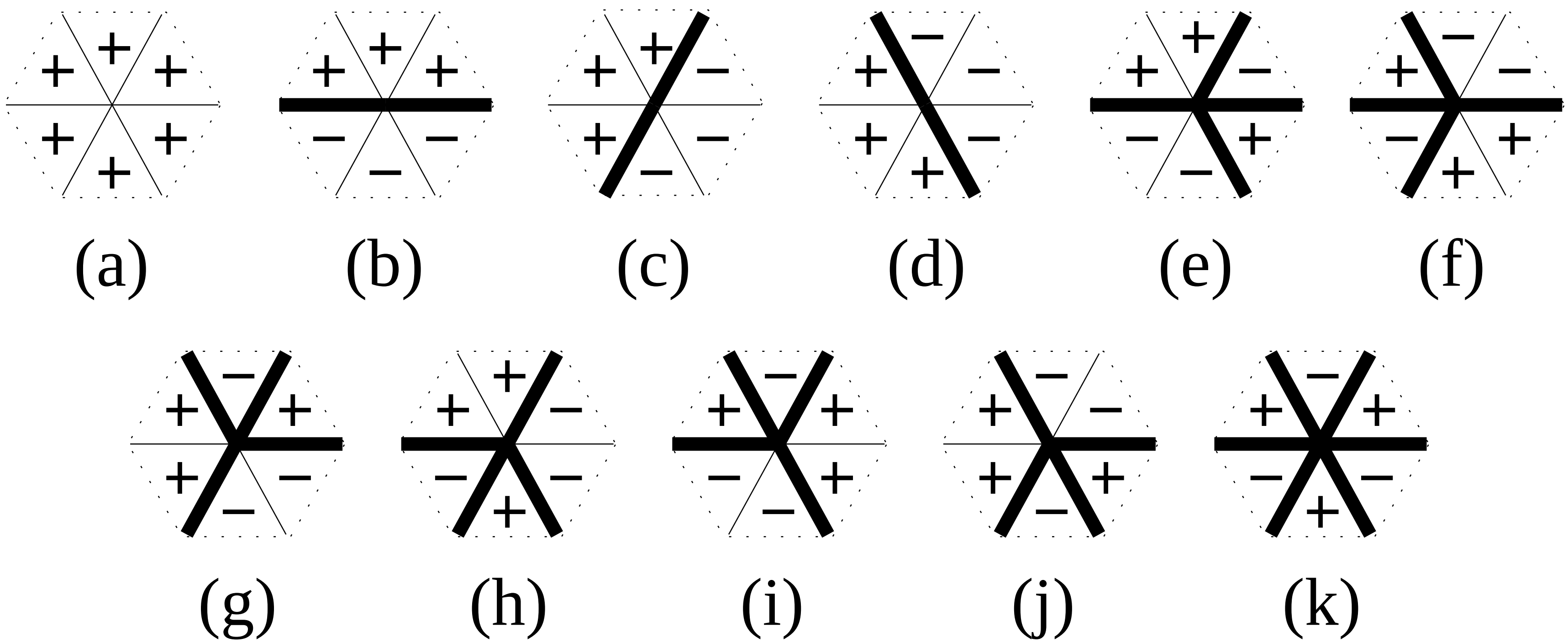
6 Bonds ----> 64 States ?

Only 11 Physical States


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)

Face Up $- - -- > S = +1$

Face Down $- - - > S = -1$

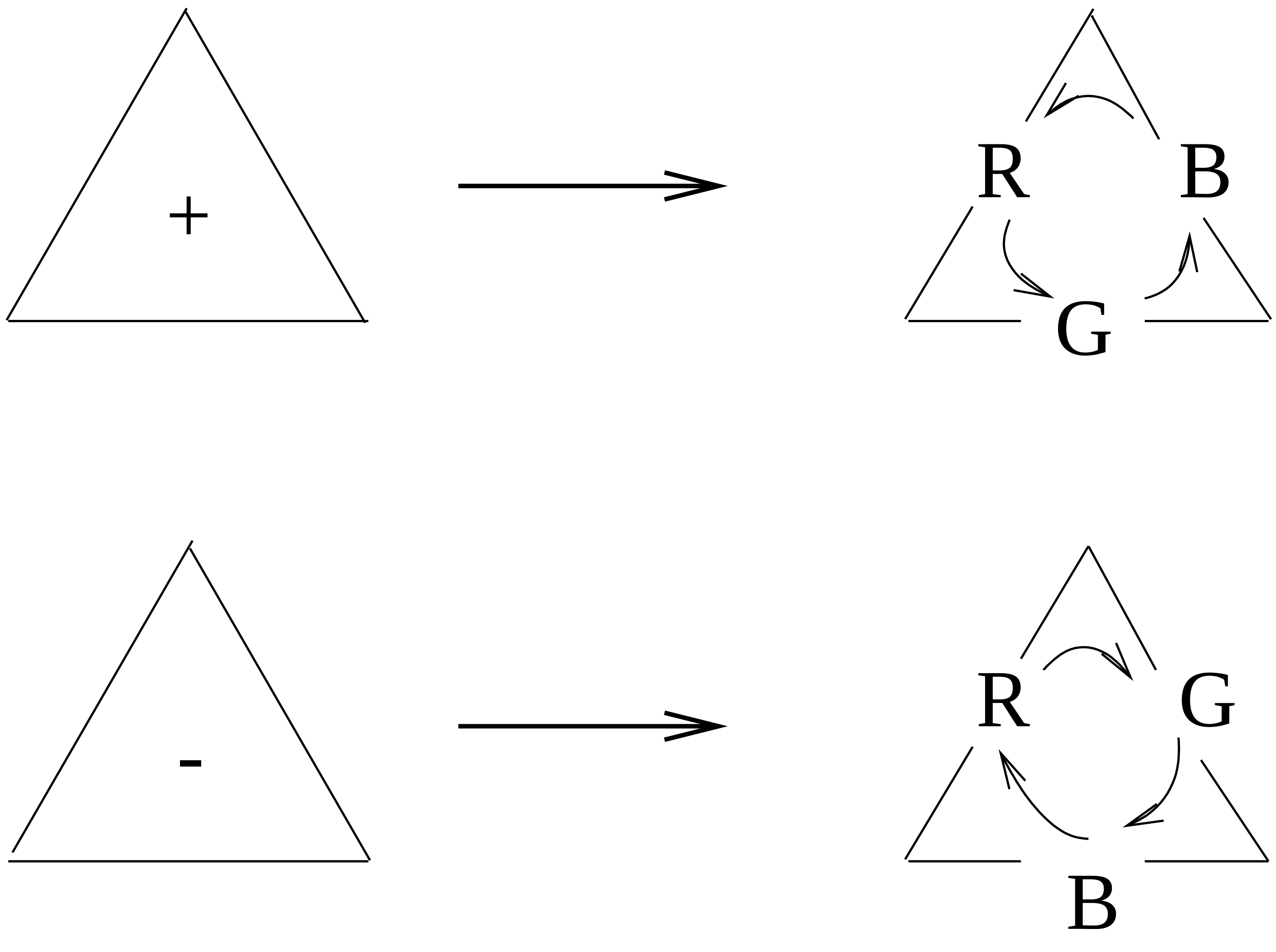


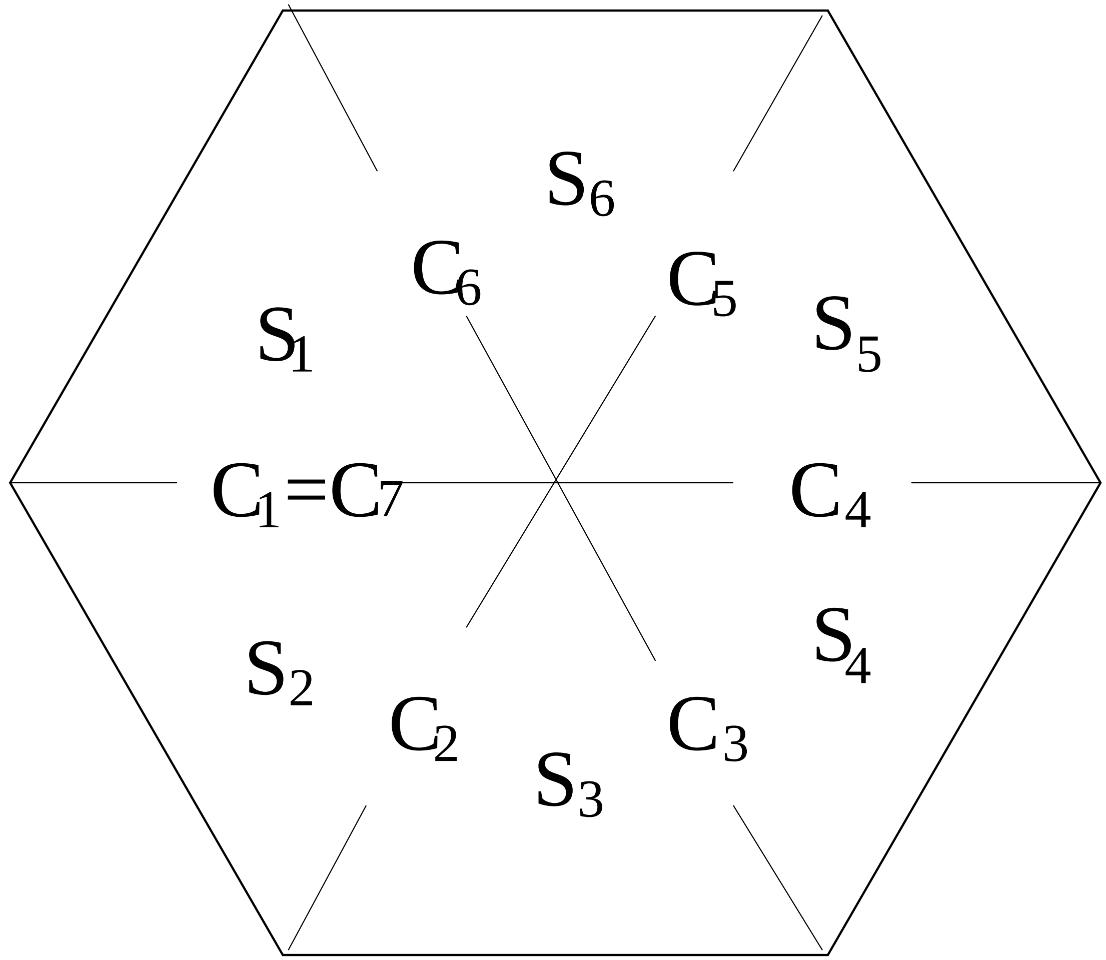
Folding Constraints (or Geometric Constraints)

$$\sigma_v \equiv \sum_{i\ around\ v} S_i = 0 \bmod 3 \tag{1}$$

Three-Coloring Formulation

R (0), G (1),B (2)





$$C_{i+1} = C_i + S_i \mod 3$$

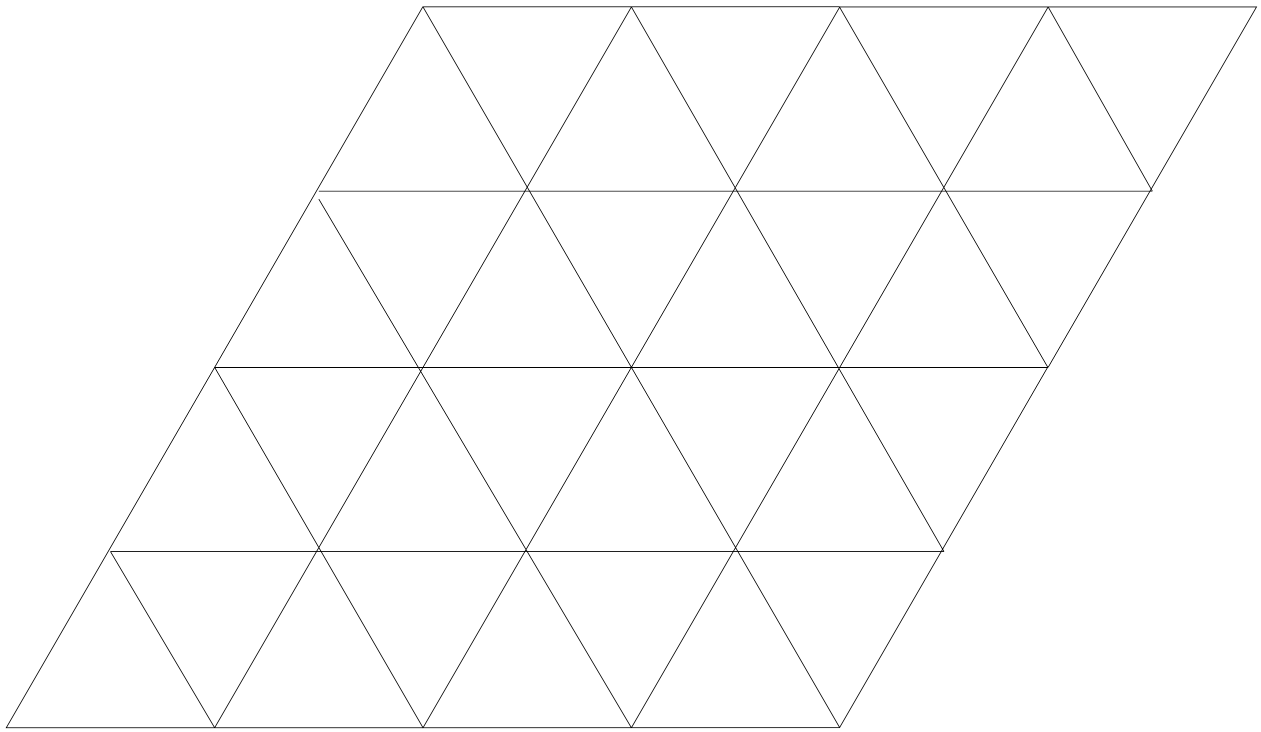
After One Turn,

$$C_7 = C_1$$

reduces to

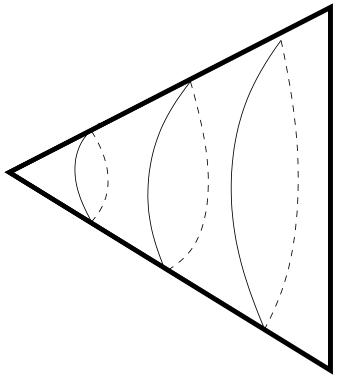
$$\sigma_v \equiv \sum_{i\ around\ v} S_i = 0 \text{ mod } 3$$

Regular Triangular Lattice



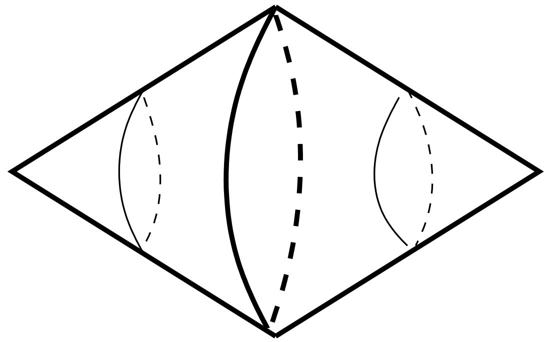
Eulerian Randomly Triangulated Surface

2 Faces



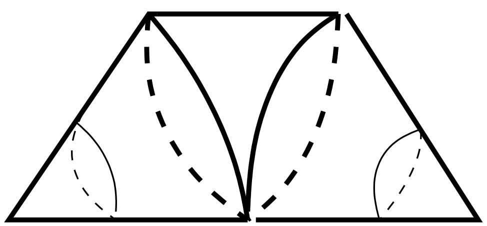
1

4 Faces

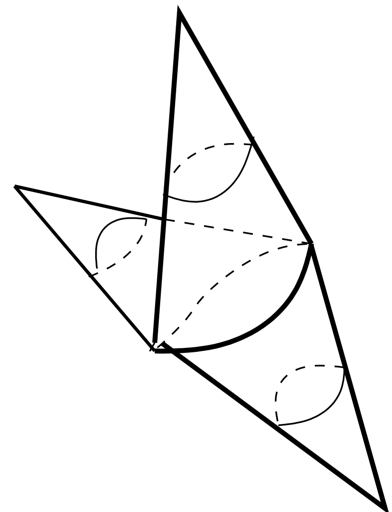


3

6 Faces

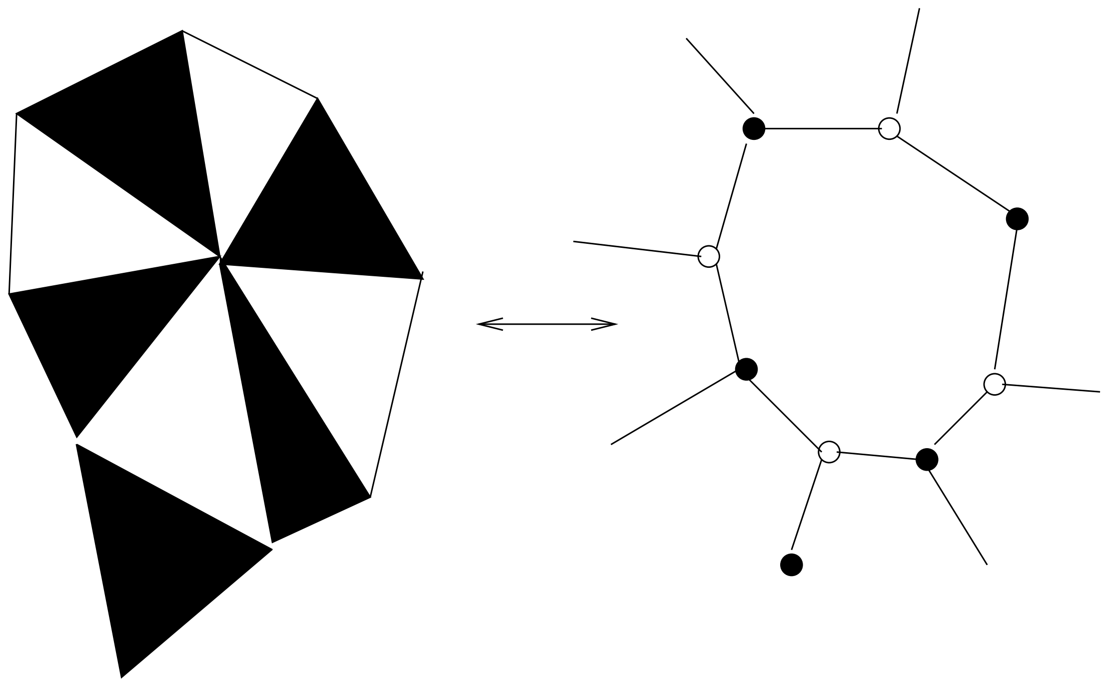


9



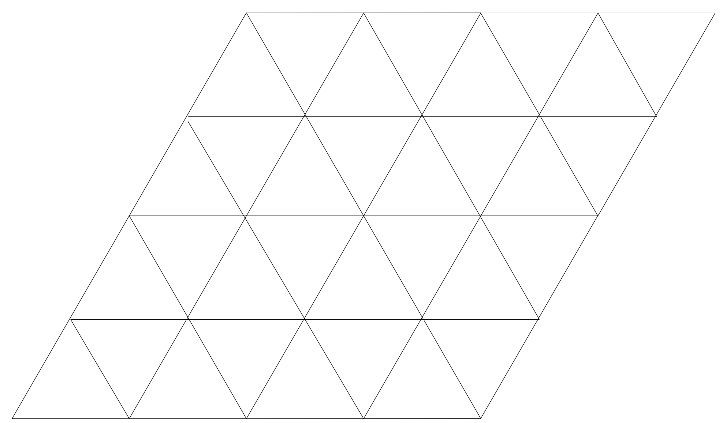
3

Foldability = Eulerian (Genus = 0)



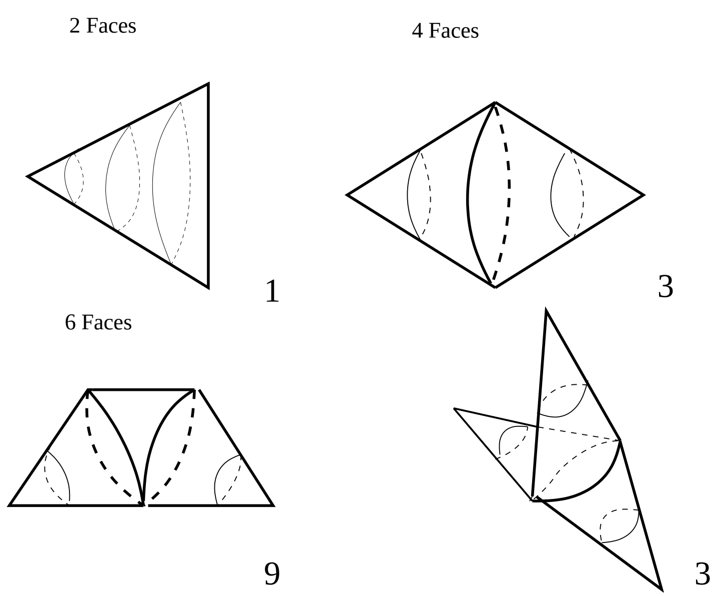
Toy Model for Polymer(ized) Membrane:

- Folding of Triangular Lattice
- Constrained Z_2 Spin System on the Dual of Triangular Lattice
- 3-Coloring Problem of Triangular Lattice



Toy Model for Fluid Membrane:

- Folding of Eulerian Randomly Triangulated Surface
- Constrained Z_2 Spin System on its Dual Random Diagram
- 3-Coloring Problem of the Random Diagram



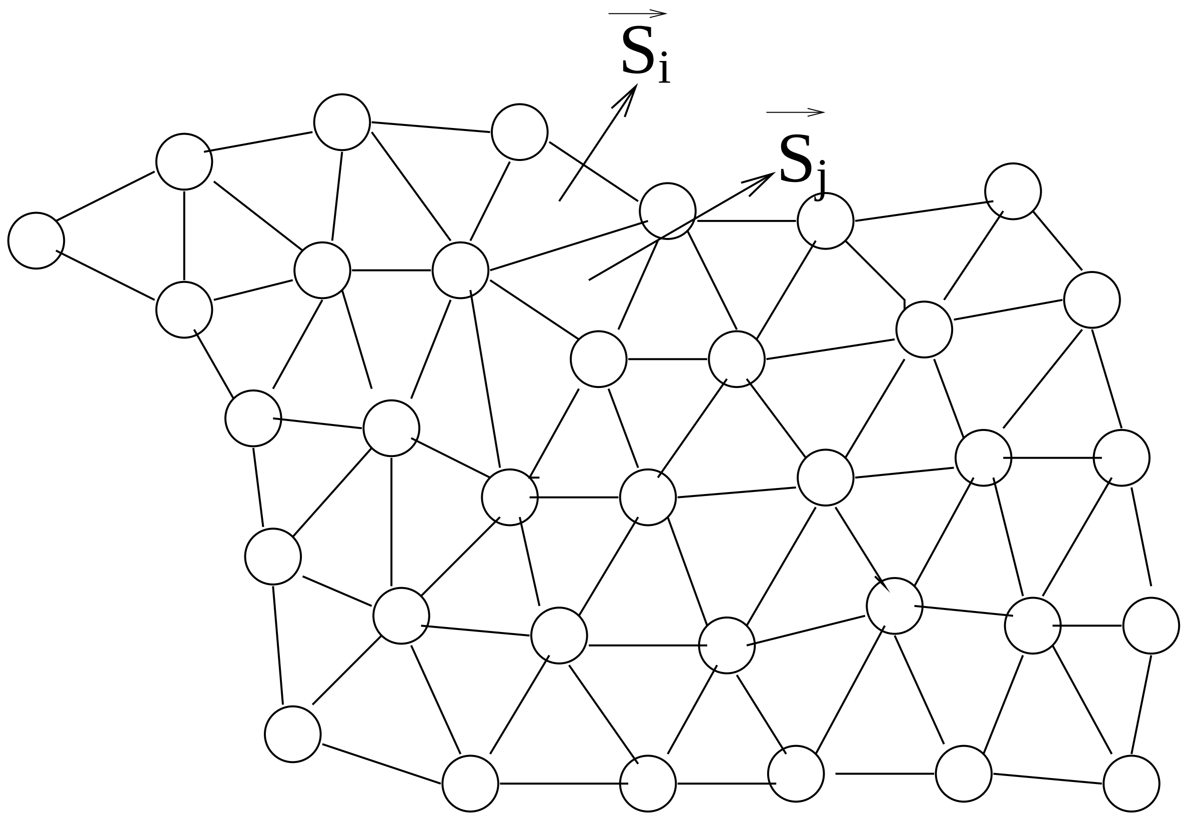
How to Formulate.

- Matrix Model ?

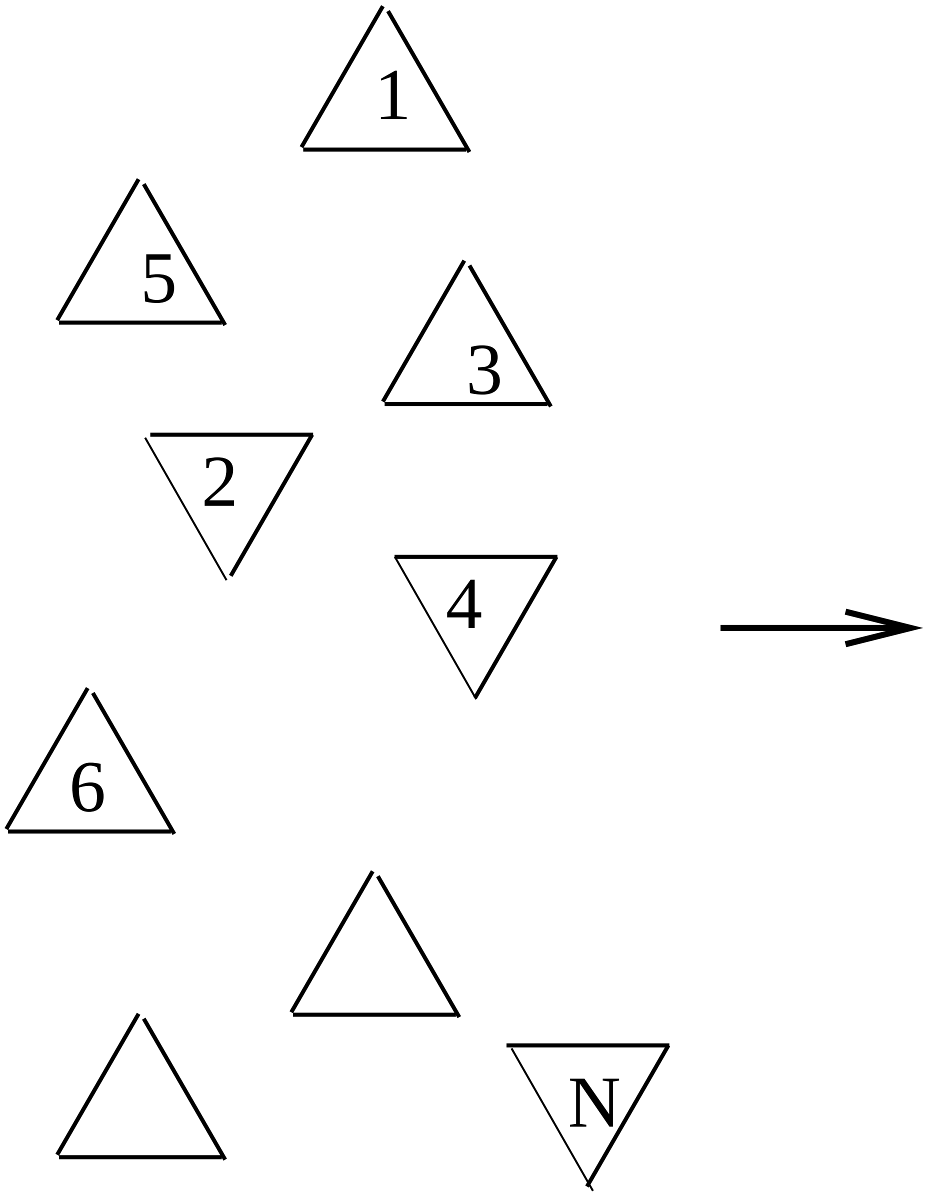
  Difficult and not yet solved
- Decorated Tree

  also difficult and under progress
- Mean Field (Cluster Variation) Approach

2　ランダム三角形分割された膜の折り畳み問題と相転移



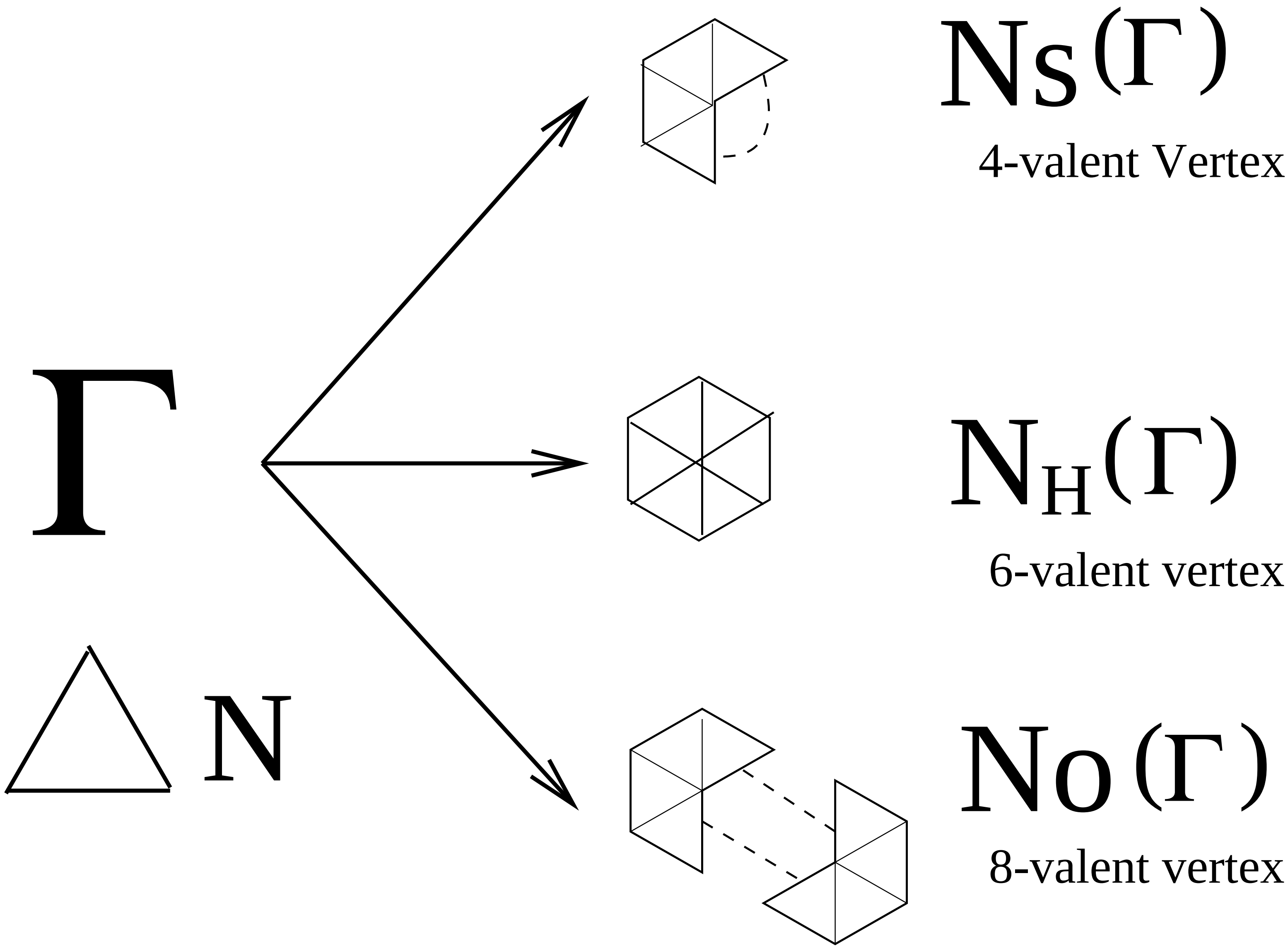
$$-\beta\mathcal{H} = K \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \quad (2)$$



Eulerian Random

Triangulated Lattice

$$\Gamma$$



Ns(Γ)
4-valent Vertex
Γ
N_H(Γ)
6-valent vertex
N
No(Γ)
8-valent vertex

Folding of $\Gamma \quad \rightarrow \quad i = S_1, S_2, \cdots, S_N$

with the Geometric Constraints

$$\sigma_v \equiv \sum_{i \;\; around \;\; v} S_i = 0 \text{ mod } 3$$

Hamiltonian

$$-\beta\mathcal{H}(i|\Gamma) = K \sum_{<ij>} S_i S_j + h \sum_i S_i \qquad (3)$$

$$K = \beta J = J/k_B T \quad \text{and} \quad h = \beta H \qquad (4)$$

N_B : Number of Spin-Pairs

$$N_B = \frac{3}{2} N$$

From Euler's Relation,

$$N_S + N_H + N_O \simeq \frac{1}{2} N = N_F$$

and the Relation

$$\frac{1}{2}(4 \times N_S + 6 \times N_H + 8 \times N_O) = N_B$$

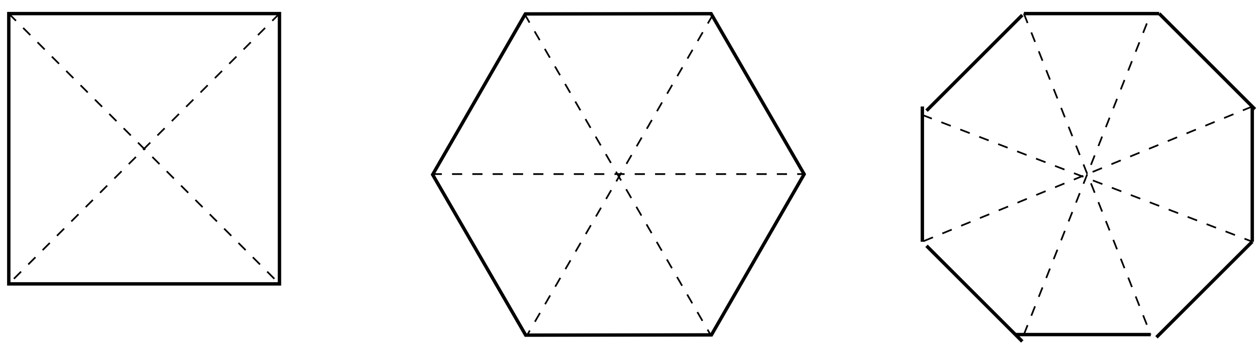
We obtain

$$N_S = N_O$$

Parameters α, β, γ:

$$\alpha(\Gamma) = \frac{N_S(\Gamma)}{N_F} \quad \beta(\Gamma) = \frac{N_H(\Gamma)}{N_F} \quad \gamma(\Gamma) = \frac{N_O(\Gamma)}{N_F}$$

$$\alpha(\Gamma) = \gamma(\Gamma) \text{ and } \beta(\Gamma) = 1 - 2\alpha(\Gamma)$$



N_S α N_H β N_O γ

Probability Distribution

$$P(i|\Gamma) = \frac{1}{Z(\Gamma)} e^{-\beta \mathcal{H}(i|\Gamma)} \quad \text{and} \quad Z(\Gamma) = \mathbf{Tr_i} e^{-\beta \mathcal{H}(i|\Gamma)}$$

$$i = (S_1, S_2, \cdots, S_N)$$

$$P(i, \Gamma) = P(i|\Gamma) P(\Gamma)$$

$$\langle A(i) \rangle = \mathbf{Tr_{i,\Gamma}} P(i, \Gamma) A(i)$$

Entropy S

$$\begin{aligned} S &= -\mathbf{Tr_{i,\Gamma}} P(i,\Gamma)\log P(i,\Gamma) \\ &= \mathbf{Tr_{i,\Gamma}}(-P(i|\Gamma)\log P(i|\Gamma))P(\Gamma) - \mathbf{Tr_{\Gamma}} P(\Gamma)\log P(\Gamma) \\ &= \mathbf{Tr_{\Gamma}}(S_{spin}[\Gamma])P(\Gamma) + S_{Lattice} \end{aligned} \tag{5}$$

Energy $\langle E \rangle$

$$\langle E \rangle = \langle \mathcal{H}(i|\Gamma) \rangle = \mathbf{Tr_{i,\Gamma}} P(i,\Gamma)\mathcal{H}(i|\Gamma)$$

Free Energy F

$$\begin{aligned} F &= \langle E \rangle - S \\ &= \mathbf{Tr_{\Gamma}}[\mathbf{Tr_{i}}(\mathcal{H}(i|\Gamma) - \log P(i|\Gamma))P(i|\Gamma) - \log P(\Gamma)]P(\Gamma) \\ &\equiv \mathbf{Tr_{\Gamma}}(F[\Gamma] - \log P(\Gamma))P(\Gamma) \end{aligned} \tag{6}$$

Cluster Variation Method = Truncation of Entropy

$$
\begin{aligned}
\frac{1}{N} S_{spin}[\Gamma] \equiv & -\frac{1}{N} \mathbf{Tr_i} P(i|\Gamma) \log P(i|\Gamma) \\
\rightarrow & -\frac{1}{2} \mathbf{Tr_1} P_{1A}(1) \log P_{1A}(1) + \frac{1}{2} \mathbf{Tr_2} P_{1B}(2) \log P_{1B}(2) \\
& -\frac{3}{2} \mathbf{Tr_{1,2}} P_2(1,2) \log P_2(1,2) \\
& +\frac{1}{2} \alpha(\Gamma) \mathbf{Tr_{1,2,3,4}} P_4 \log P_4 + \frac{1}{2} \beta(\Gamma) \mathbf{Tr_{1,2,3,4,5,6}} P_6 \log P_6 \\
& +\frac{1}{2} \gamma(\Gamma) \mathbf{Tr_{1,2,3,4,5,6,7,8}} P_8 \log P_8 \qquad (7)
\end{aligned}
$$

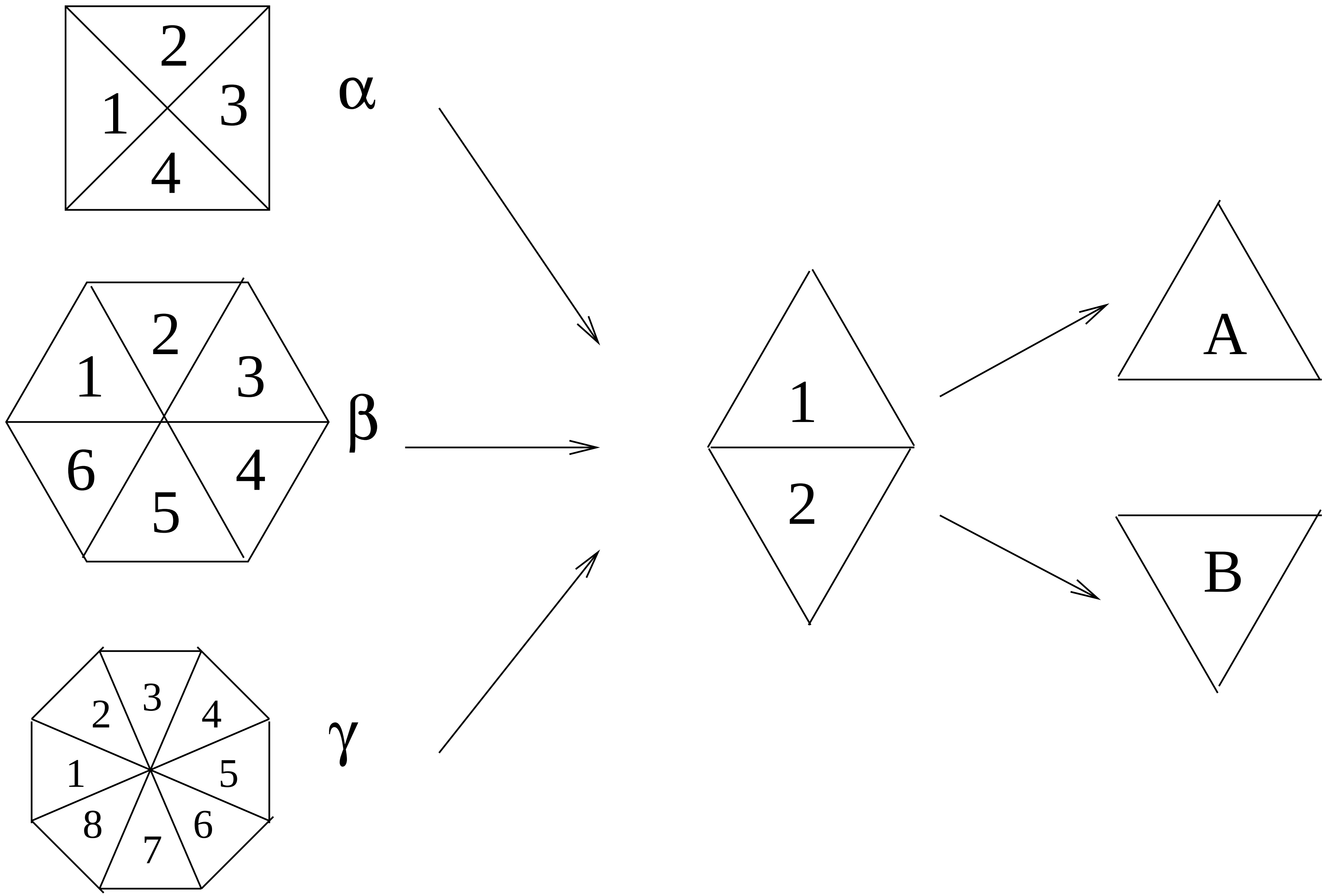
$$
\mathbf{Tr_{1,2}} \equiv \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \qquad \text{etc}
$$

$$\begin{aligned} P_2(1,2) = {} & \frac{2}{3}\alpha(\Gamma)\mathbf{STr_{3,4}}P_4(S_1,S_2,S_3,S_4) \\ & + \beta(\Gamma)\mathbf{STr_{3,4,5,6}}P_6(S_1,S_2,S_3,S_4,S_5,S_6) \\ & + \frac{4}{3}\gamma(\Gamma)\mathbf{STr_{3,4,5,6,7,8}}P_8(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8) \end{aligned} \quad (8)$$

$$P_{1A}(S_1) = \mathbf{Tr_2}P_2(S_1,S_2) \quad \text{and} \quad P_{1B}(S_2) = \mathbf{Tr_1}P_2(S_1,S_2) \quad (9)$$

Reduction from P_4, P_6, P_8 to P_2

$$\begin{aligned} \mathbf{STr_{3,4}}P_4 = {} & \frac{1}{4}\mathbf{Tr_{3,4}}(P_4(S_1,S_2,S_3,S_4) + P_4(S_3,S_2,S_1,S_4) \\ & + P_4(S_3,S_4,S_1,S_2) + P_4(S_1,S_3,S_4,S_2)) \end{aligned} \quad (10)$$


2
1
3
4
α
2
1
3
β
6
4
5
3
2
4
1
5
8
6
7
γ
1
2
A
B

$$\mathbf{STr}_{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}} P_6 = \frac{1}{6} \sum_{3,4,5,6} (P_6(S_1, S_2, S_3, S_4, S_5, S_6) + P_6(S_3, S_2, S_1, S_4, S_5, S_6)$$
$$+ P_6(S_3, S_4, S_1, S_2, S_5, S_6) + P_6(S_3, S_4, S_5, S_2, S_1, S_6)$$
$$+ P_6(S_3, S_4, S_5, S_6, S_1, S_2) + P_6(S_1, S_3, S_4, S_5, S_6, S_2)) \quad (11)$$

$$\mathbf{STr}_{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}} P_8 = \frac{1}{8} \sum_{3,4,5,6,7,8} (P_8(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8)$$
$$+ P_8(S_3, S_2, S_1, S_4, S_5, S_6, S_7, S_8) + P_8(S_3, S_4, S_1, S_2, S_5, S_6, S_7, S_8)$$
$$+ P_8(S_3, S_4, S_5, S_2, S_1, S_6, S_7, S_8) + P_8(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8)$$
$$+ P_8(S_3, S_4, S_5, S_6, S_1, S_2, S_7, S_8) + P_8(S_3, S_4, S_5, S_6, S_7, S_2, S_1, S_8)$$
$$+ P_8(S_1, S_3, S_4, S_5, S_6, S_7, S_8, S_2)) \quad (12)$$

$$
\begin{aligned}
f[\Gamma] &\equiv \frac{1}{N}F[\Gamma] \\
&= -\frac{3}{2}K\mathbf{Tr}_{\mathbf{1},\mathbf{2}}P_2(S_1,S_2)S_1S_2 - \frac{1}{2}h\mathbf{Tr}_{\mathbf{1}}P_{1A}(S_1)S_1 - \frac{1}{2}h\mathbf{Tr}_{\mathbf{2}}P_{1B}(S_2)S_2 \\
&\quad -\frac{1}{N}S_{spin}[\Gamma] \\
&\quad +\lambda_8(\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}}P_8(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8)-1) \\
&\quad +\lambda_6(\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}}P_6(S_1,S_2,S_3,S_4,S_5,S_6)-1) \\
&\quad +\lambda_4(\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}}P_4(S_1,S_2,S_3,S_4)-1) \qquad (13)
\end{aligned}
$$

Natural Iteration Method (or Variational Equation)

$$P_N(S_1, S_2, \cdots, S_N) = \exp(-\lambda_N + \frac{K}{2}\sum_{i=1}^{N} S_i S_{i+1} + \frac{h}{3}\sum_{i=1}^{N} S_i)$$
$$\times \ (P_2(S_1, S_N) P_2(S_1, S_2) P_2(S_3, S_2) \cdots P_2(S_{N-1}, S_N))^{\frac{1}{2}}$$
$$\times \ (P_{1A}(S_1) P_{1B}(S_2) \cdots P_{1A}(S_{N-1}) P_{1B}(S_N))^{-\frac{1}{3}} \quad (14)$$

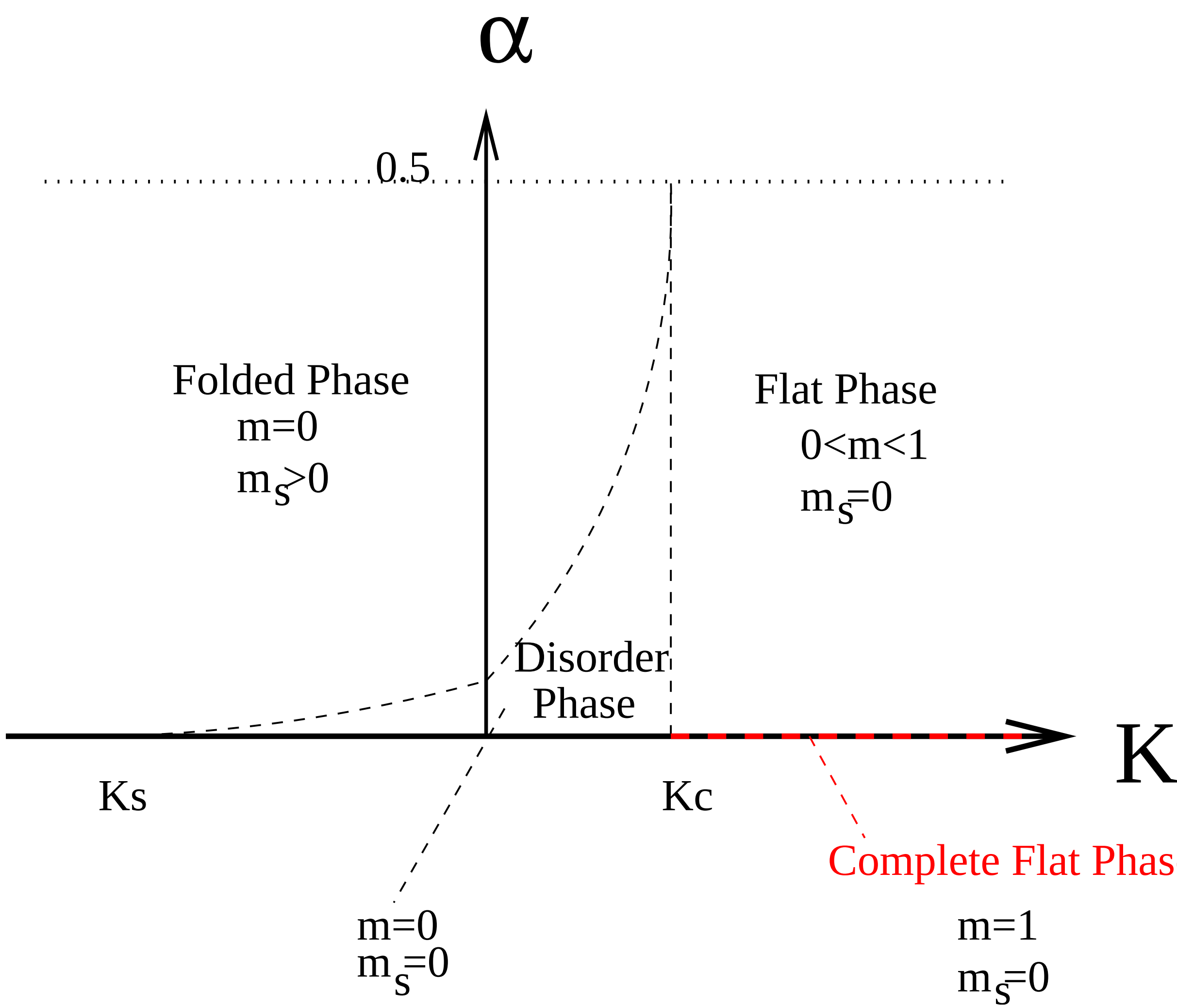
with

$$S_{N+1} = S_1$$

and

$$N = 4, 6, 8$$

Phase Diagram in (K, α)-Plane

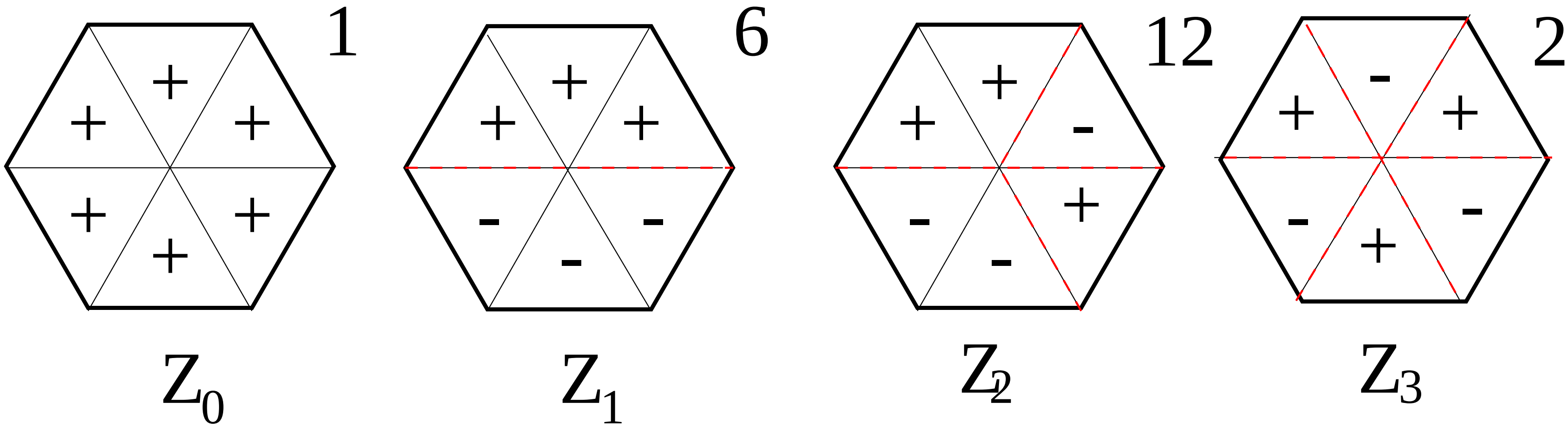


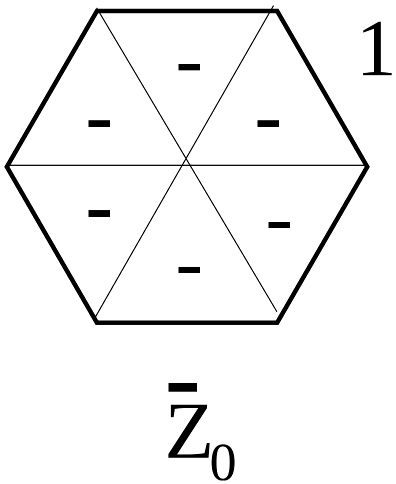
Regular Triangular Lattice Case ($\beta = 1$)

Confi g.	M	M_S	deg.	Ferro. S.B.	Antiferro S.B.
$++++++$	6	0	1	Z_0	Z_0
$------$	-6	0	1	$\overline{Z_0}$	Z_0
$+++---$	0	2	3	Z_1	Z_1
$---+++$	0	-2	3	Z_1	$\overline{Z_1}$
$+--++-$	0	2	6	Z_2	Z_2
$-++--+$	0	-2	6	Z_2	$\overline{Z_2}$
$+-+-+-$	0	6	1	Z_3	Z_3
$-+-+-+$	0	-6	1	Z_3	$\overline{Z_3}$

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$





Ferromagnetic S.B.

Ferromagnetic S.B. Case :

Introducing x, y as

$$x = \frac{Z_1}{\sqrt{Z_0 \overline{Z_0}}} \equiv \frac{Z_1}{W} \quad \text{Fugacity per Fold}$$

$$y = \frac{Z_0}{\overline{Z_0}} \quad \text{Order Parameter} \tag{15}$$

$$Z_0 = W y^{\frac{1}{2}} \quad \overline{Z_0} = W y^{-\frac{1}{2}} \quad Z_1 = W x \quad Z_2 = W x^2 \quad Z_3 = W x^3$$

Algebraic Relation between x and y as

$$y = \left(\frac{y^{\frac{1}{2}} + 2x + 2x^2}{y^{\frac{-1}{2}} + 2x + 2x^2} \right)^3 \left(\frac{y^{\frac{-1}{2}} + 3x + 6x^2 + x^3}{y^{\frac{1}{2}} + 3x + 6x^2 + x^3} \right)^2 \tag{16}$$

$$x = e^{-2K} \frac{(x + 4x^2 + x^3)}{\left((y^{\frac{1}{2}} + 2x + 2x^2)(y^{\frac{-1}{2}} + 2x + 2x^2) \right)^{\frac{1}{2}}} \tag{17}$$

Solutions:

- $y = 1$: Disordered Phase

$$x = \frac{2 - u + \sqrt{3 - u - u^2}}{2u - 1} \quad \text{with} \quad u = e^{2K}$$

$$x = 2 \qquad c =< S_1 S_2 >= -\frac{1}{3} \text{ at } K = 0$$

- $x = 0$ and $y =$arbitrary : Completely Flat Phase

Anti-Ferromagnetic S.B. Case :

Introducing x, y as

$$x = \frac{\sqrt{Z_1 \overline{Z_1}}}{Z_0} \quad \text{Fugacity per Fold}$$

$$y = \frac{Z_1}{\overline{Z_1}} \quad \text{Order Parameter} \tag{18}$$

$$\begin{aligned} Z_1 &= W y^{\frac{1}{2}} x & \overline{Z_1} &= W y^{-\frac{1}{2}} \\ Z_2 &= W y^{\frac{1}{2}} x^2 & \overline{Z_2} &= W y^{-\frac{1}{2}} x^2 \\ Z_3 &= W y^{\frac{3}{2}} x^3 & \overline{Z_3} &= W y^{-\frac{3}{2}} x^3 \end{aligned} \tag{19}$$

Algebraic Relation between x and y as

$$y = \frac{xy^{\frac{1}{2}} + 3x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}} + x^3y^{\frac{3}{2}}}{xy^{-\frac{1}{2}} + 3x^2y^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^3y^{-\frac{3}{2}}} \left(\frac{1 + 2xy^{-\frac{1}{2}} + xy^{\frac{1}{2}} + 2x^2y^{\frac{1}{2}} + 4x^2y^{-\frac{1}{2}} + x^3y^{-\frac{3}{2}}}{1 + 2xy^{\frac{1}{2}} + xy^{-\frac{1}{2}} + 2x^2y^{-\frac{1}{2}} + 4x^2y^{\frac{1}{2}} + x^3y^{\frac{3}{2}}} \right)$$

(20

$$x = e^{-2K} \frac{(xy^{\frac{1}{2}} + 3x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}} + x^3y^{\frac{3}{2}})(xy^{-\frac{1}{2}} + 3x^2y^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^3y^{-\frac{3}{2}})}{1 + xy^{\frac{1}{2}} + xy^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}}}$$

(21

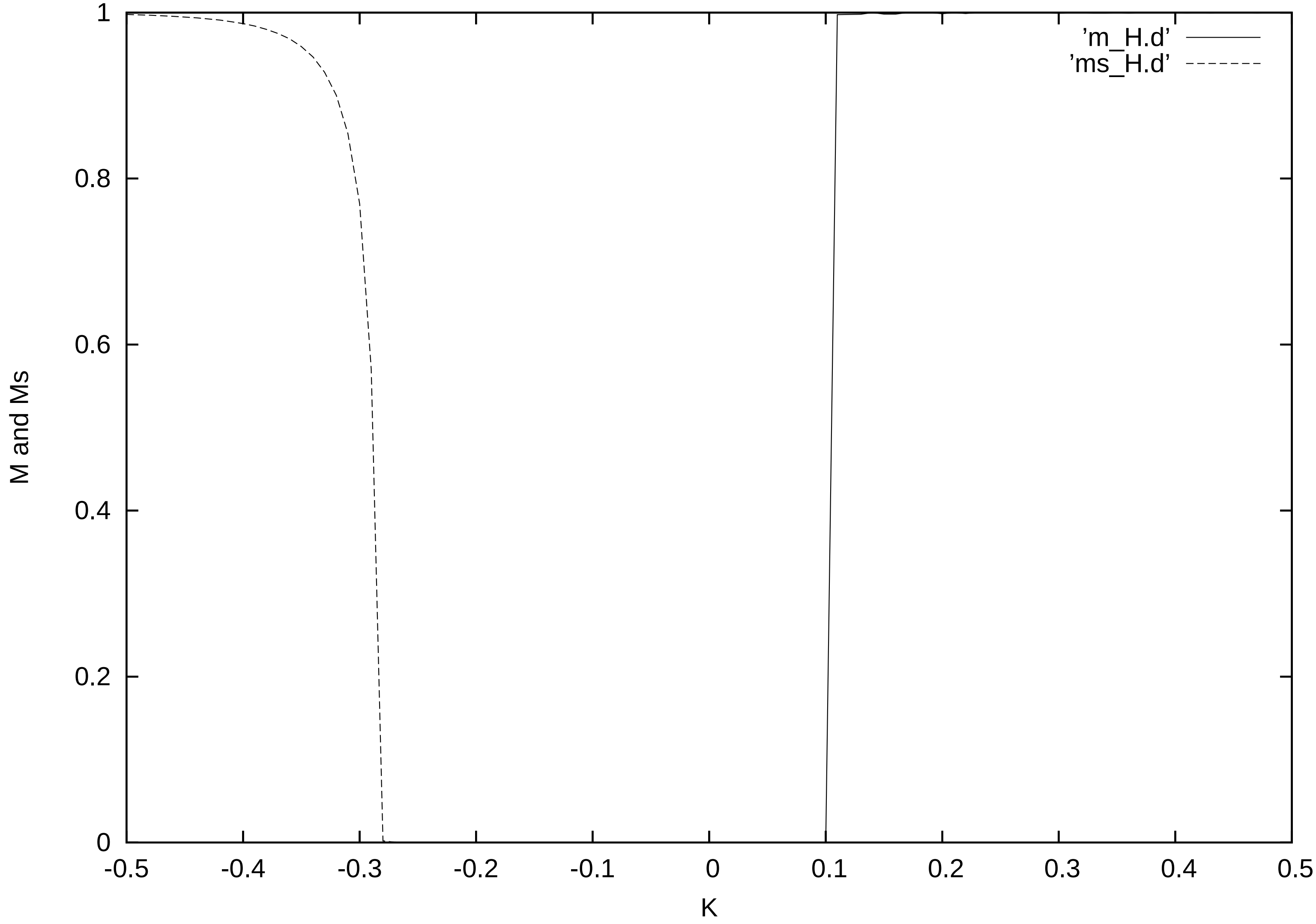
Solutions:

- $y = 1$: Disordered Phase
- $y = 1 + \epsilon$: Folded (Antiferromagnetic) Phase

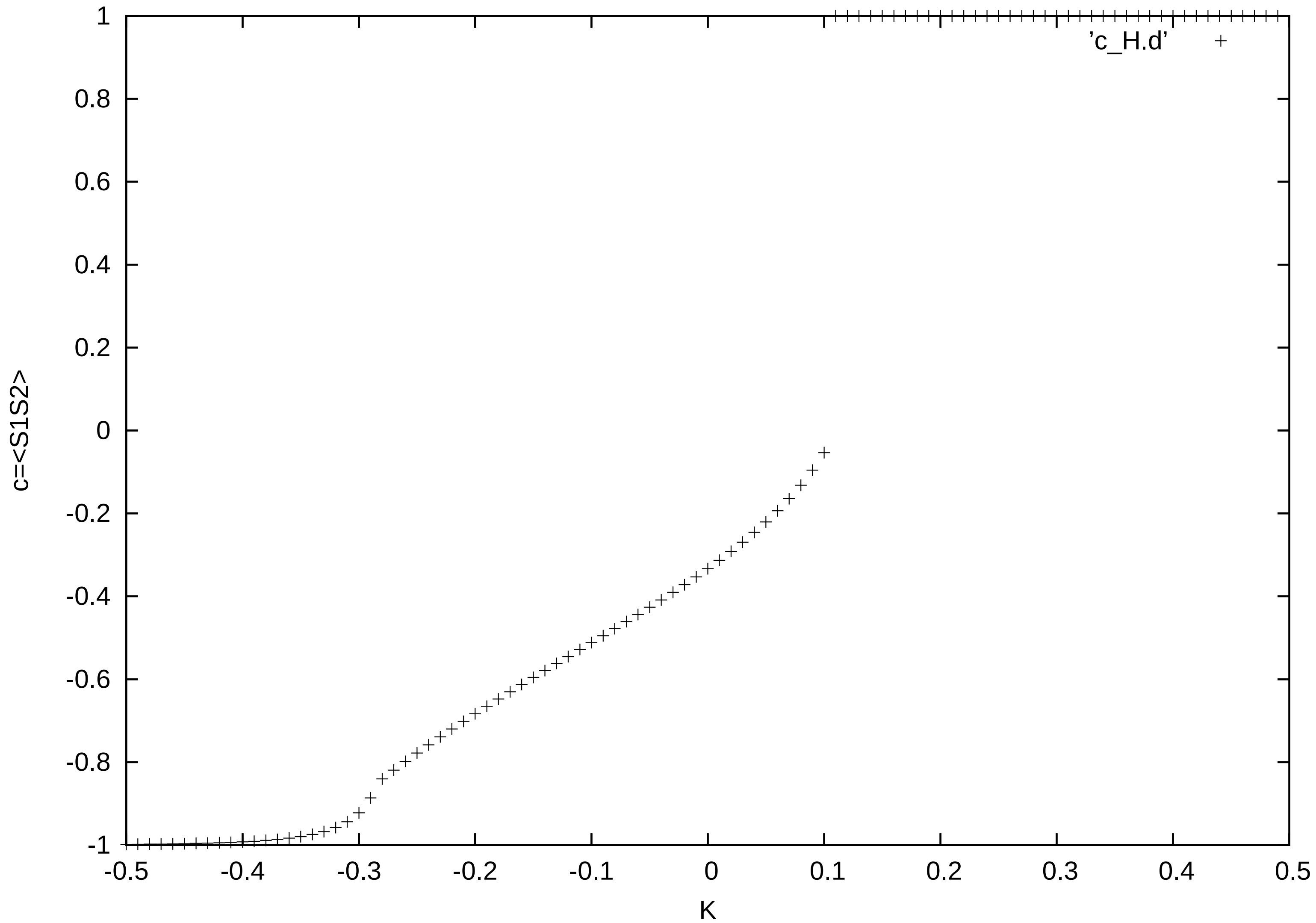
$$x_{st}^3 - 21x_{st}^2 - 12x_{st} - 4 = 0$$

$$K_{st} = -0.2838\cdots$$

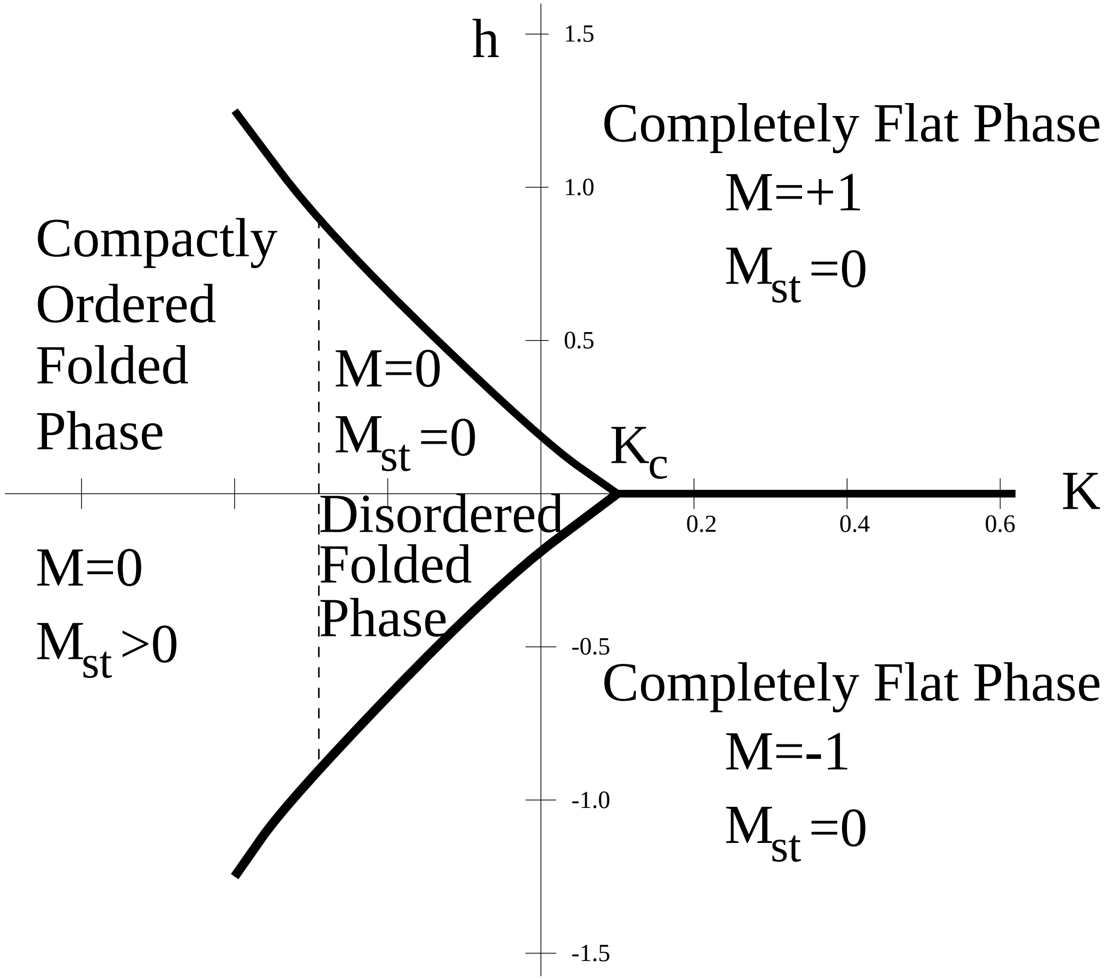
Phase Transitions



$c = < S_1 S_2 >$



Phase Diagram:

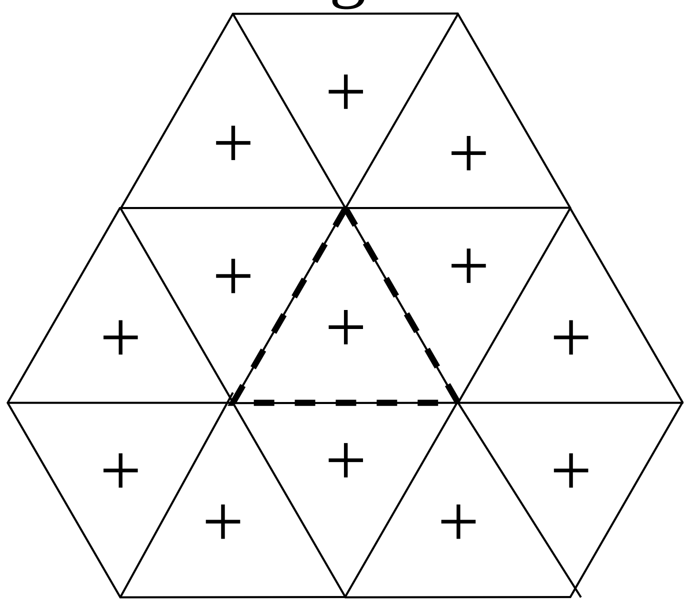


M.Cirillo, G.Gonella and A.Pelizzola ('95)

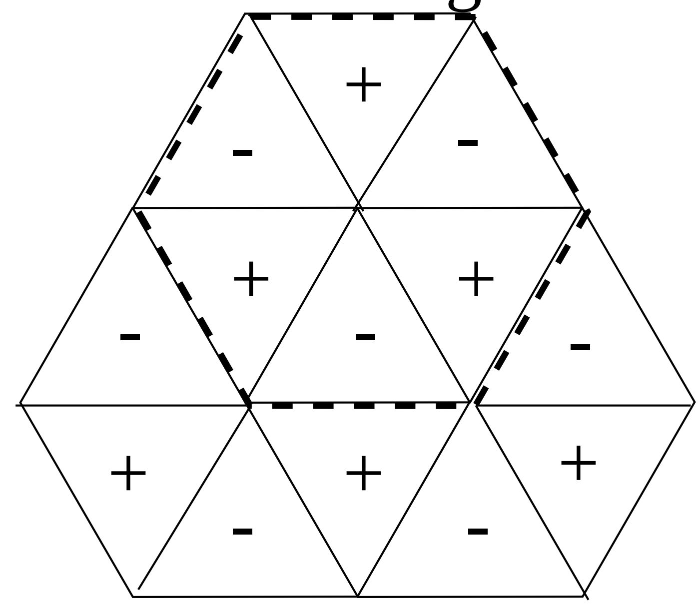
Ph. Di Francesco, E.Guitter and S.Mori ('96)

Local Excitation

Flat Config.



Piled Config.



Local Spin Flip

+ ---> -

- ---> +

Impossible

Possible

Exact Results

Entropy or Number of States N_S for N Triangles:

$$N_S \sim q^N > 2^{\frac{1}{6} \times N}$$

$$q = \frac{\sqrt{3}}{2\pi}\Gamma(1/3)^{\frac{3}{2}} = 1.208717\cdots$$

Ph. Di Francesco and E.Guitter ('94)

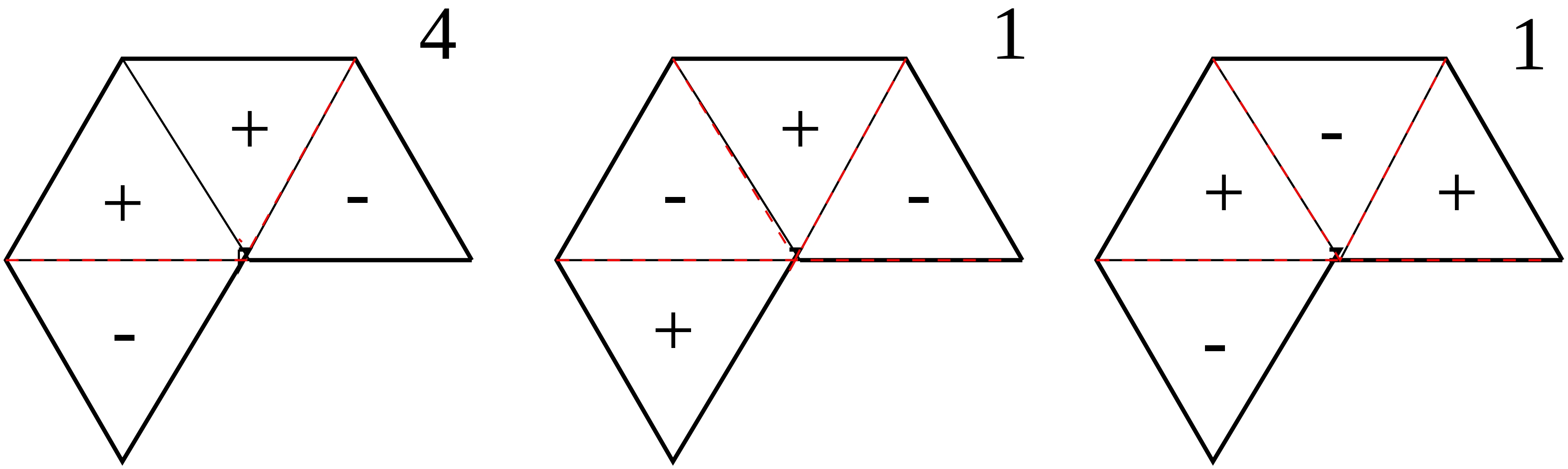
$$q_{CVM} = 1.2019$$

Pure 4-valent Veretx Case ($\alpha = 1$)

Confi g.	M	M_S	deg.	Antiferro S.B.
$++--$	0	0	4	Z_0
$+-+-$	0	4	1	Z_1
$-+-+$	0	-4	1	$\overline{Z_1}$

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$



Z_0 Z_1 $\bar{Z}_1$

AntiFerromagnetci S.B.

Introducing x, y as

$$x = \frac{\sqrt{Z_1 \overline{Z_1}}}{Z_0} \equiv \frac{Z_1}{W} \quad \text{Fugacity per Fold}$$
$$y = \frac{Z_1}{\overline{Z_1}} \quad \text{Order Parameter} \tag{22}$$

$$Z_1 = Wxy^{\frac{1}{2}} \quad \overline{Z_1} = Wxy^{-\frac{1}{2}}$$

$$y = \frac{(1 + xy^{\frac{1}{2}})^2 (2 + xy^{-\frac{1}{2}})^{\frac{4}{3}}}{(1 + xy^{-\frac{1}{2}})^2 (2 + xy^{\frac{1}{2}})^{\frac{4}{3}}} \tag{23}$$
$$x = e^{-2K} (1 + xy^{\frac{1}{2}})(1 + xy^{-\frac{1}{2}}) \tag{24}$$

Solutions:

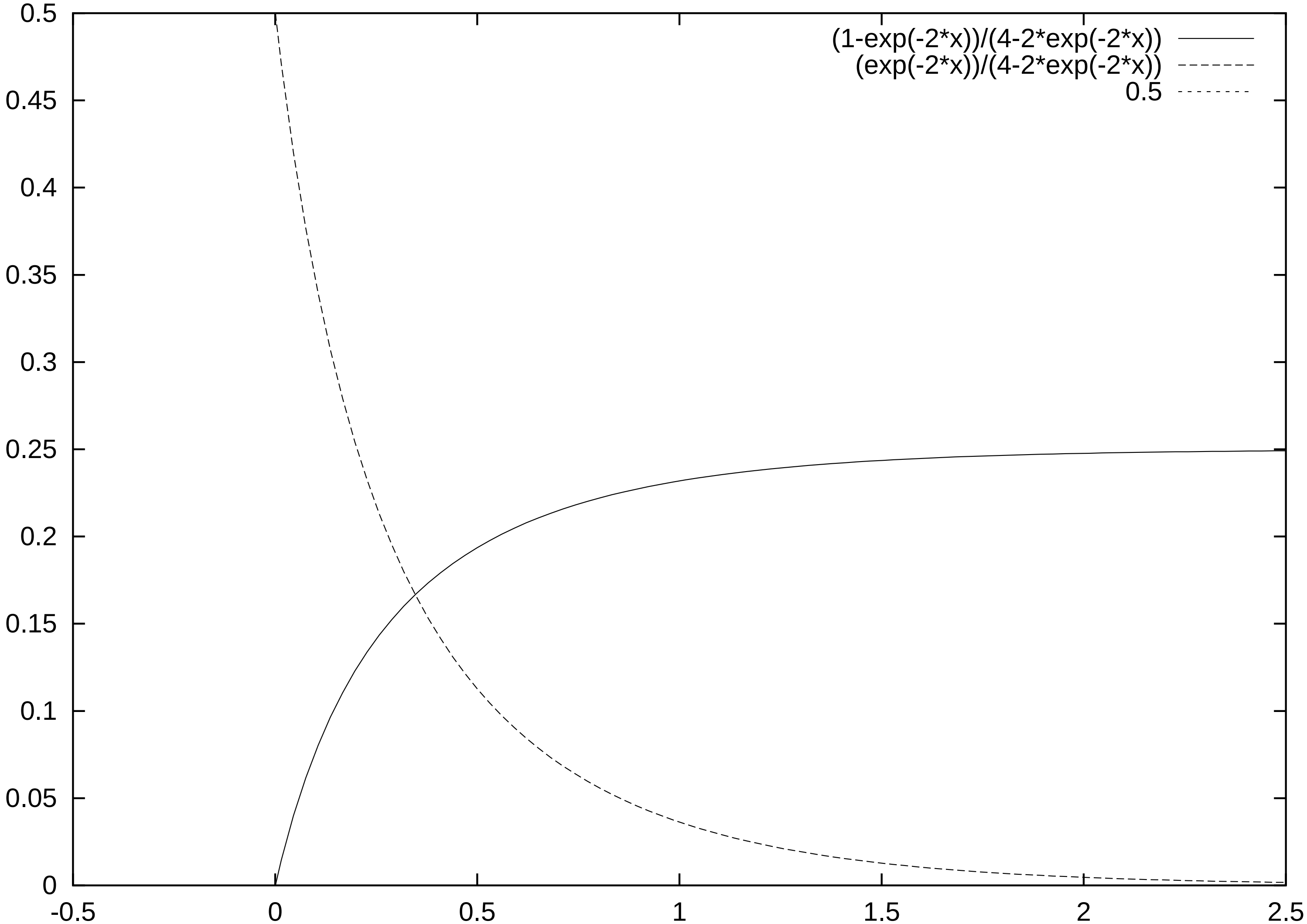
- $y = 1$: Disordered Phase

$$
\begin{aligned}
x &= \frac{e^{-2K}}{1 - e^{-2K}} \\
Z_0 &= \frac{1 - e^{-2K}}{4 - 2e^{-2K}} \\
Z_1 &= \overline{Z_1} = \frac{e^{-2K}}{4 - 2e^{-2K}} \\
c &= < S_1 S_2 >= -2 \times Z_1
\end{aligned} \tag{25}
$$

- $y = 1 + \epsilon$ does not have Real Solution for K

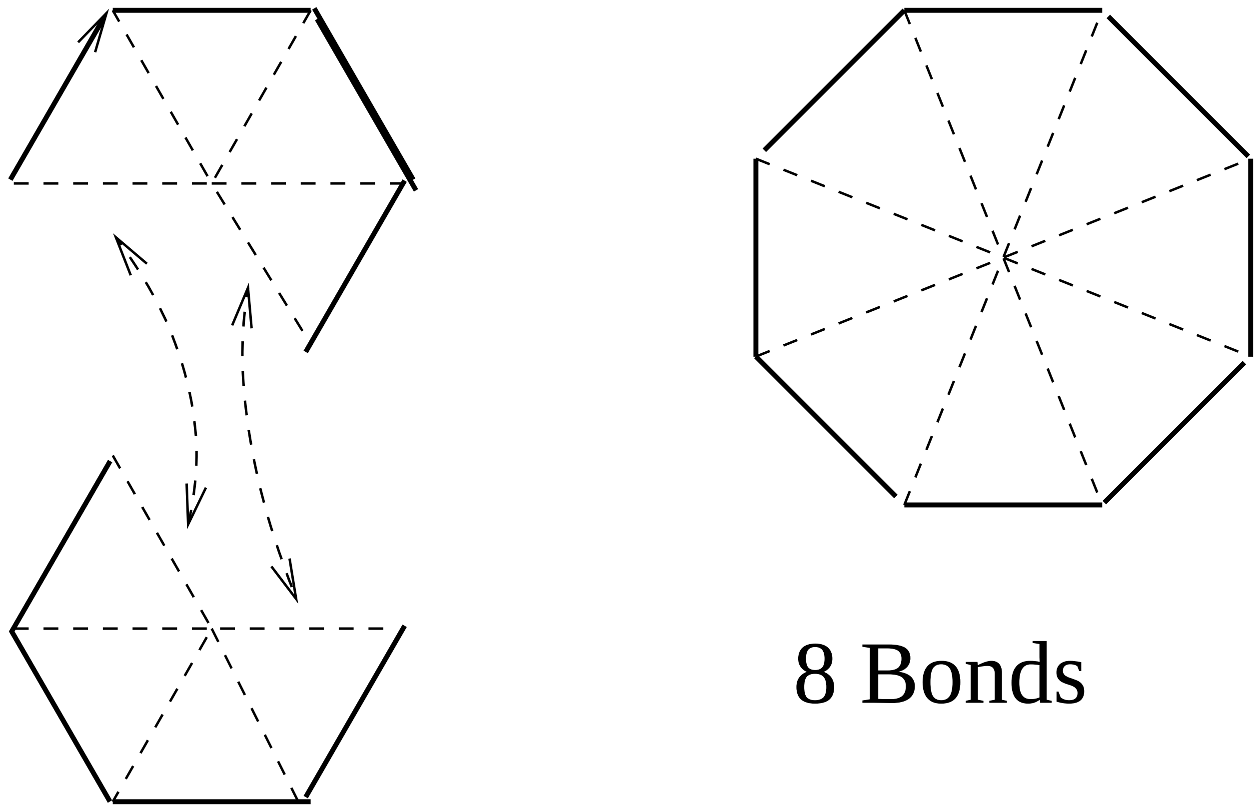
No Folded (Antiferromagnetic) Phase. Always Disorderd.

Z_0 and Z_1


(1-exp(-2*x))/(4-2*exp(-2*x))
(exp(-2*x))/(4-2*exp(-2*x))
0.5
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.5
0
0.5
1
1.5
2
2.5

Pure 8-Valent Vertex Case ($\gamma = 1$)

Foldable



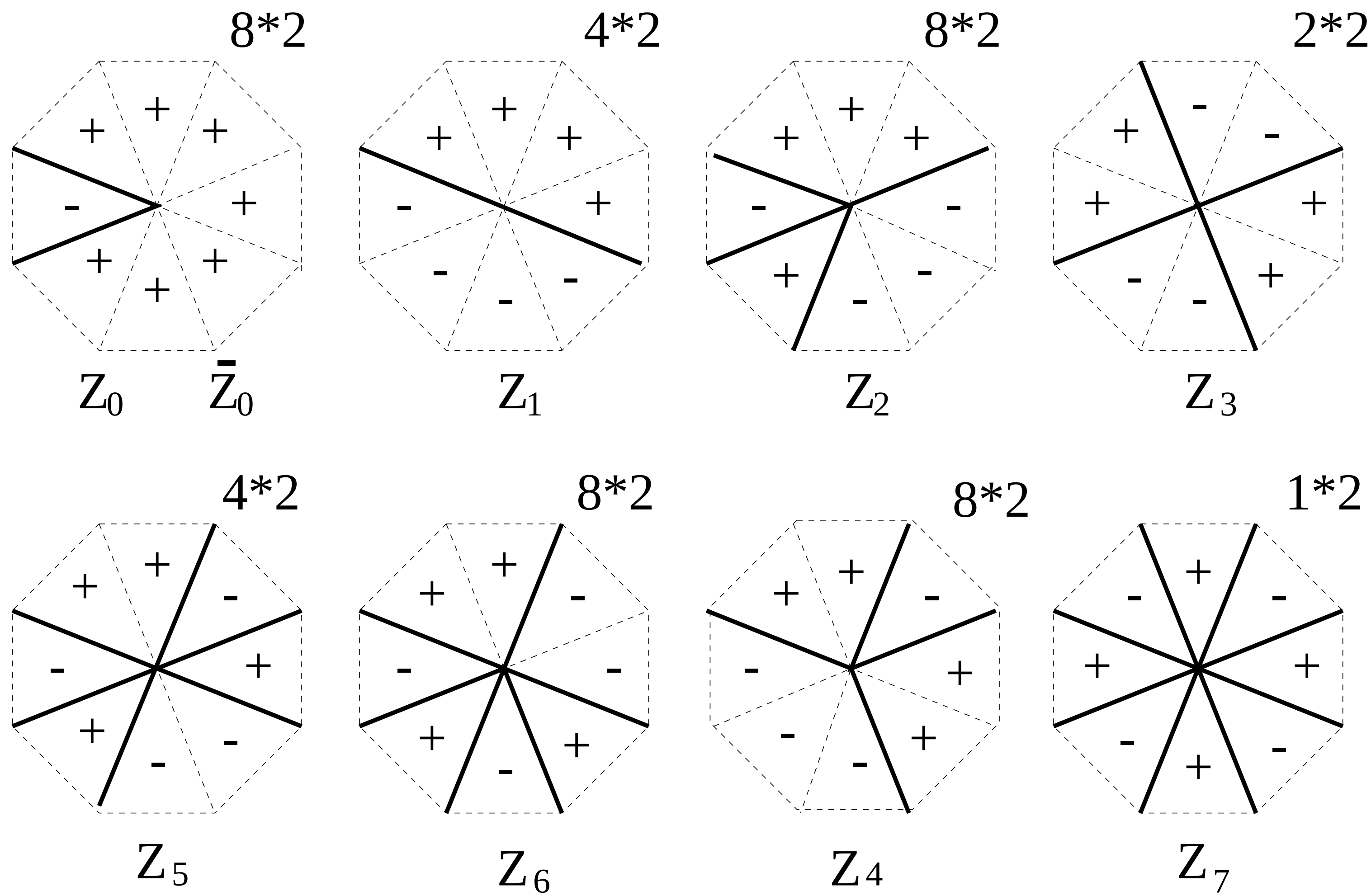
8 Bonds

43 States

Confi g.	M	deg.	Ferro. S.B.
$++++++++-$	6	8	Z_0
$+-------- $	-6	8	$\overline{Z_0}$
$++++----$	0	8	Z_1
$+++---+-$	0	16	Z_2
$++--++--$	0	4	Z_3
$+--+++--$	0	16	Z_4
$+-++-+--$	0	8	Z_5
$++-+-+--$	0	16	Z_6
$+-+-+-+-$	0	2	Z_7

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$



Ferromagnetic S.B.

Ferromagnetic S.B. Case :

Introducing x, y as

$$x = \frac{Z_2}{\sqrt{Z_0 \overline{Z_0}}} \equiv \frac{Z_2}{W} \quad \text{Fugacity per Fold}$$

$$y = \frac{Z_0}{\overline{Z_0}} \quad \text{Order Parameter} \tag{26}$$

$$Z_0 = W y^{\frac{1}{2}} \quad \overline{Z_0} = W y^{-\frac{1}{2}}$$

$$Z_1 = W \quad Z_2 = W x$$

$$Z_3 = W x \quad Z_4 = W x$$

$$Z_5 = W x^2 \quad Z_6 = W x^2$$

$$Z_7 = W x^3 \tag{27}$$

Algebraic Relations between x and y

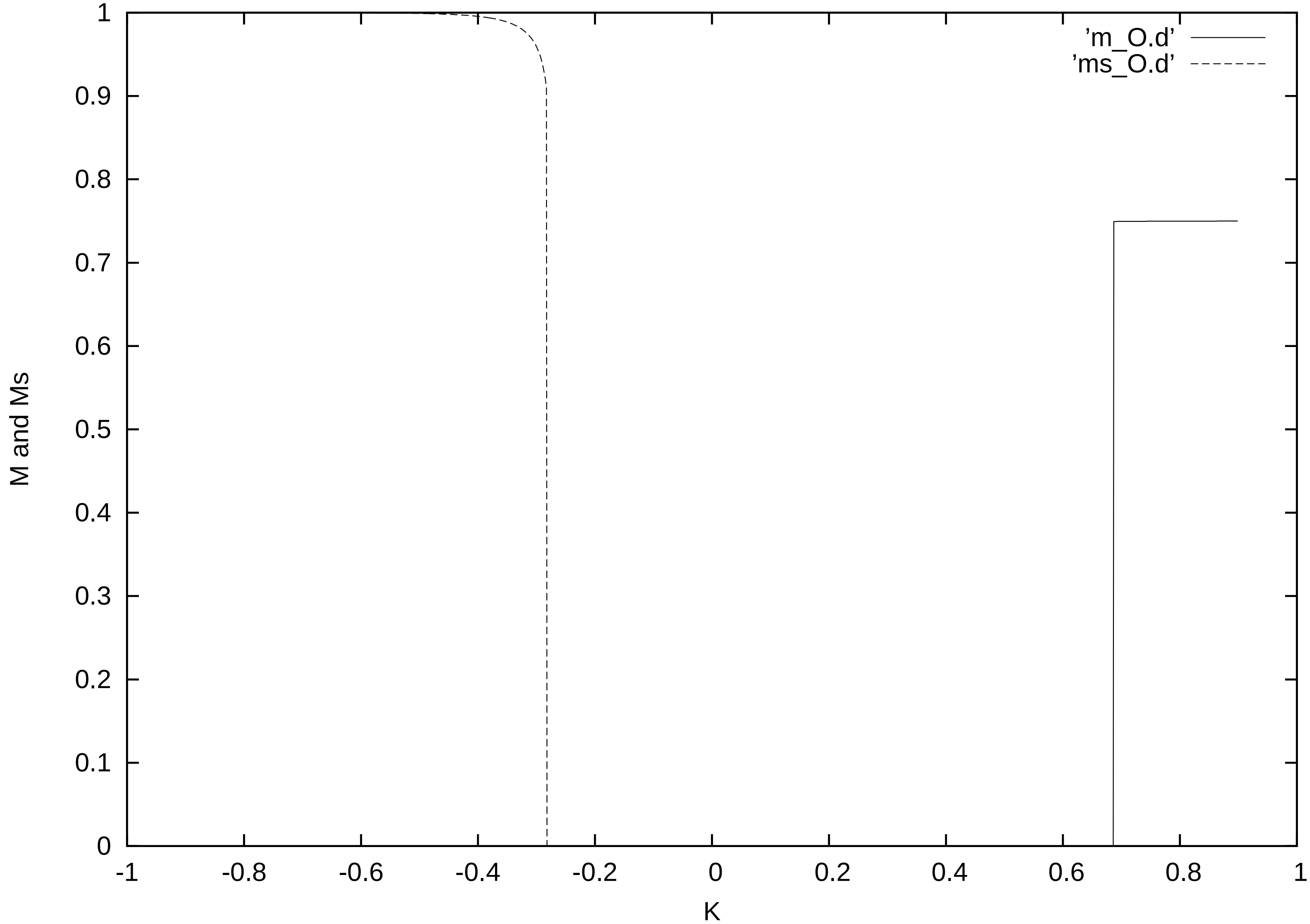
$$y = \left(\frac{4x^2 + 9x + 6y^{\frac{1}{2}} + 3}{4x^2 + 9x + 6y^{-\frac{1}{2}} + 3}\right)^3 \left(\frac{x^3 + 12x^2 + 18x + 7y^{\frac{1}{2}} + y^{-\frac{1}{2}} + 4}{x^3 + 12x^2 + 18x + 7y^{-\frac{1}{2}} + y^{\frac{1}{2}} + 4}\right)^2$$

$$x = \frac{1}{u} \times \frac{x^3 + 8x^2 + 9x + y^{-\frac{1}{2}} + y^{\frac{1}{2}} + 1}{(4x^2 + 9x + 6y^{\frac{1}{2}} + 3)^{\frac{1}{2}}(4x^2 + 9x + 6y^{-\frac{1}{2}} + 3)^{\frac{1}{2}}} \qquad (28)$$

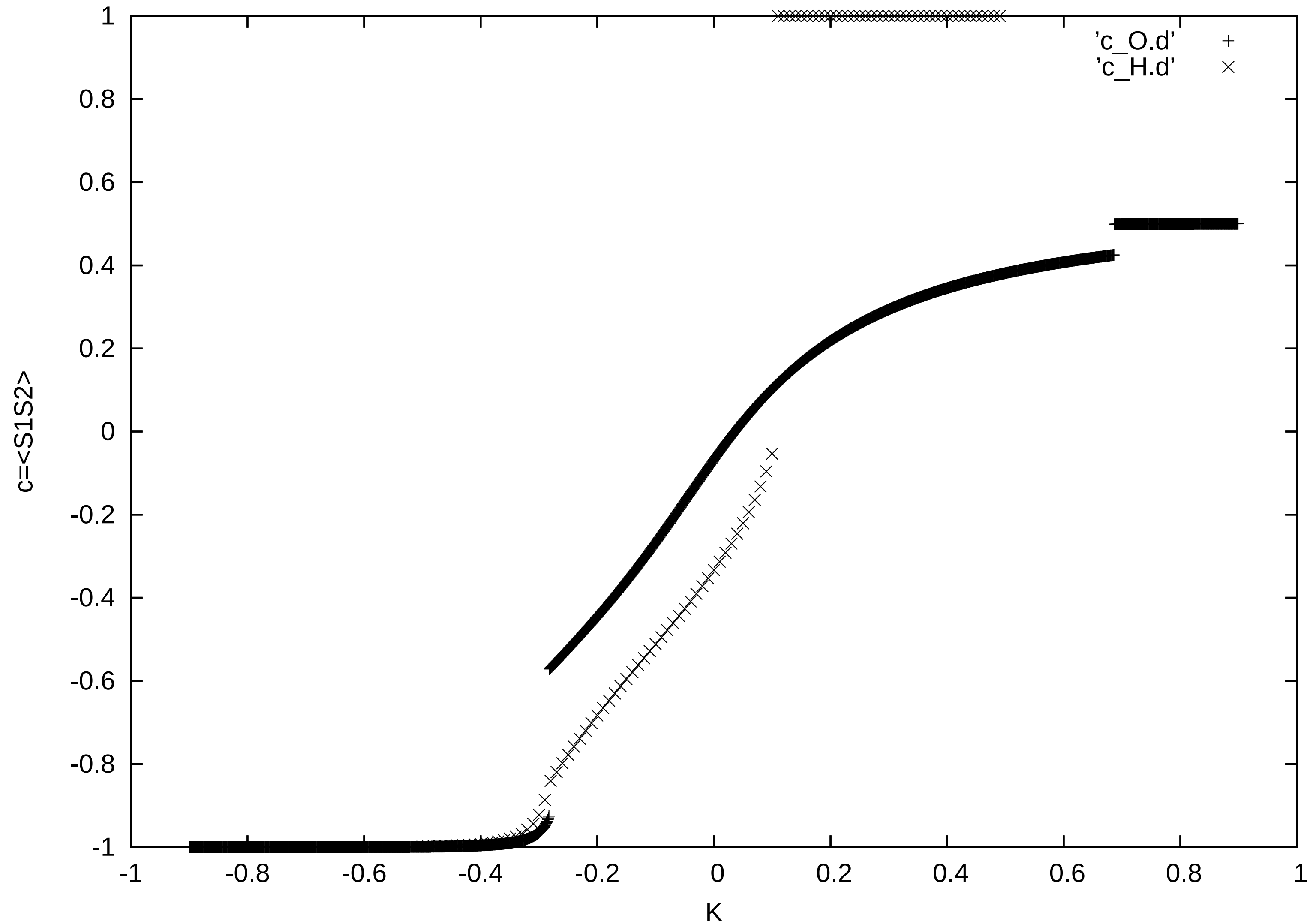
Disorderd Phase $(y = 1)$

$$x = \frac{1}{u}\frac{3 + 9x + 8x^2 + x^3}{9 + 9x + 4x^2}$$

Phase Transitions

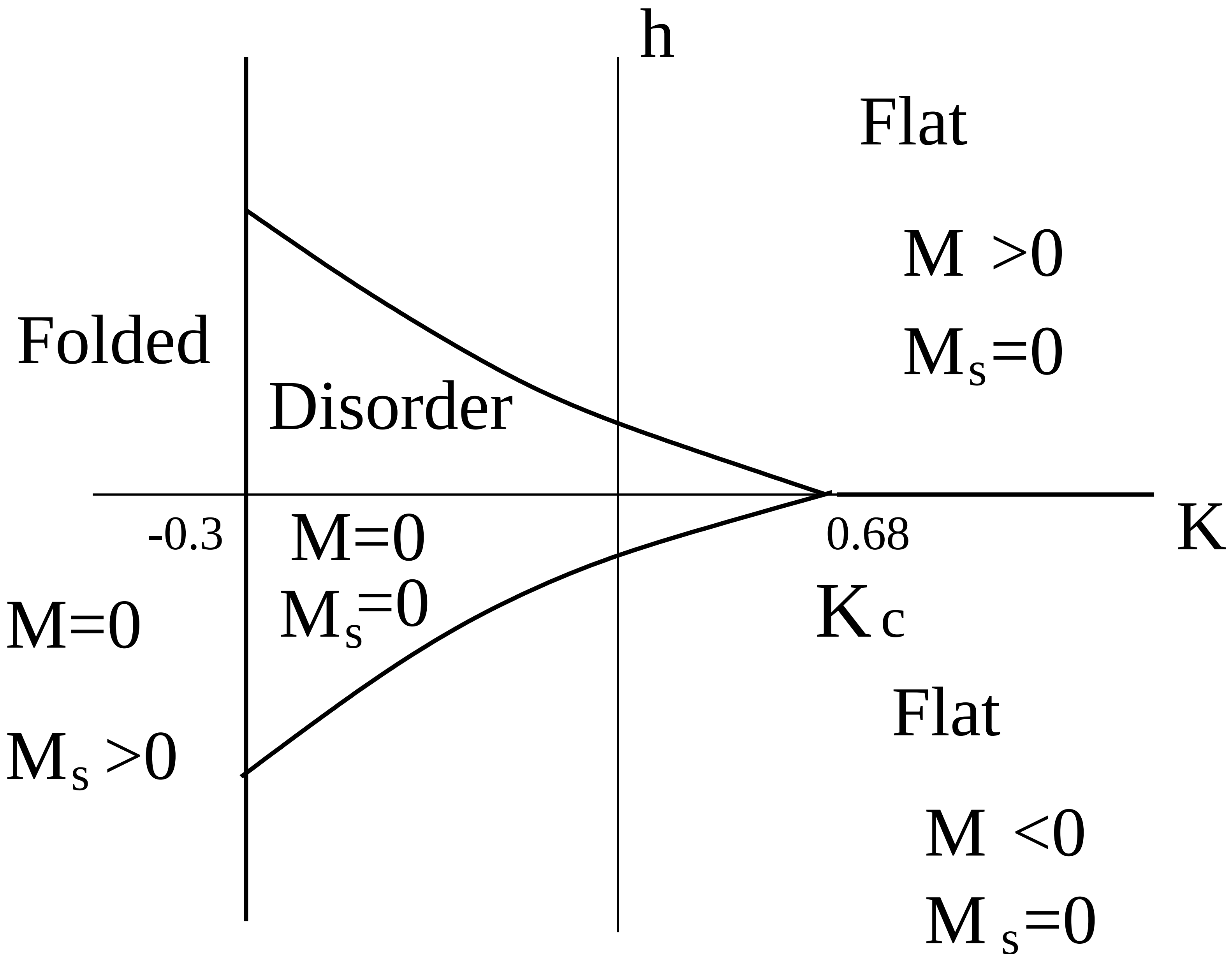


$$c = < S_1 S_2 >$$

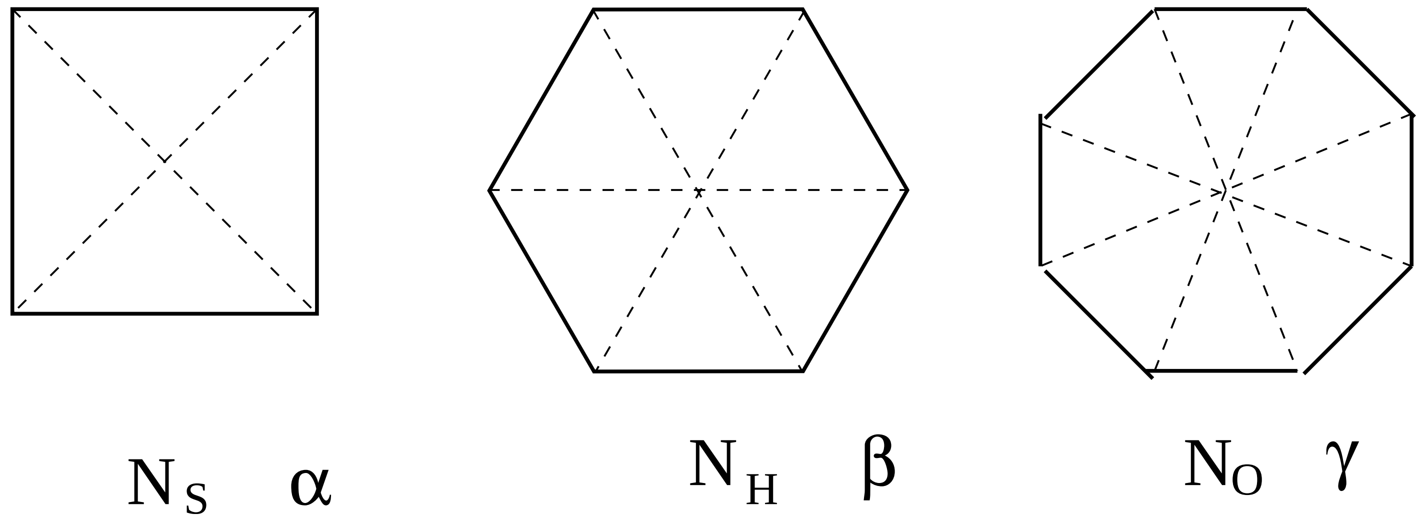


'c_O.d'
'c_H.d'
c=<S1S2>
K
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Phase Diagram



General Case ($\alpha = \gamma$)



- 4-valent Vertex $\rightarrow$ (Strong) Anti-Ferro.
- 6-valent Vertex $\rightarrow$ Not-Frustrated, (Weak) Anti-Ferro.
- 8-valent Vertex $\rightarrow$ Frustrated

Algebraic Relations for the Disordered Phase ($M = M_S = 0$)

Introducing x (Fugacity per Fold) as

$$x = \frac{1}{u} \frac{P_2(1,-1)}{P_2(1,1)}$$

All weights are written with Z_0^S, Z_0^H, Z_0^O and x as

$$\begin{aligned}
&Z_1^S = Z_0^S x \\
&Z_1^H = Z_0^H x \quad Z_2^H = Z_0^H x^2 \quad Z_3^H = Z_0^H x^3 \\
&Z_1^O = Z_0^O \quad Z_2^O = Z_0^O x \quad Z_3^O = Z_0^O x \quad Z_4^O = Z_0^O x \\
&Z_5^O = Z_0^O x^2 \quad Z_6^O = Z_0^O x^2 \quad Z_7^O = Z_0^O x^3
\end{aligned} \tag{29}$$

with

$$Z_0^S(4+2x) = 1 \quad Z_0^H(2+6x+12x^2+2x^3) = 1 \quad Z_0^O(24+36x+24x^2+2) = 1$$

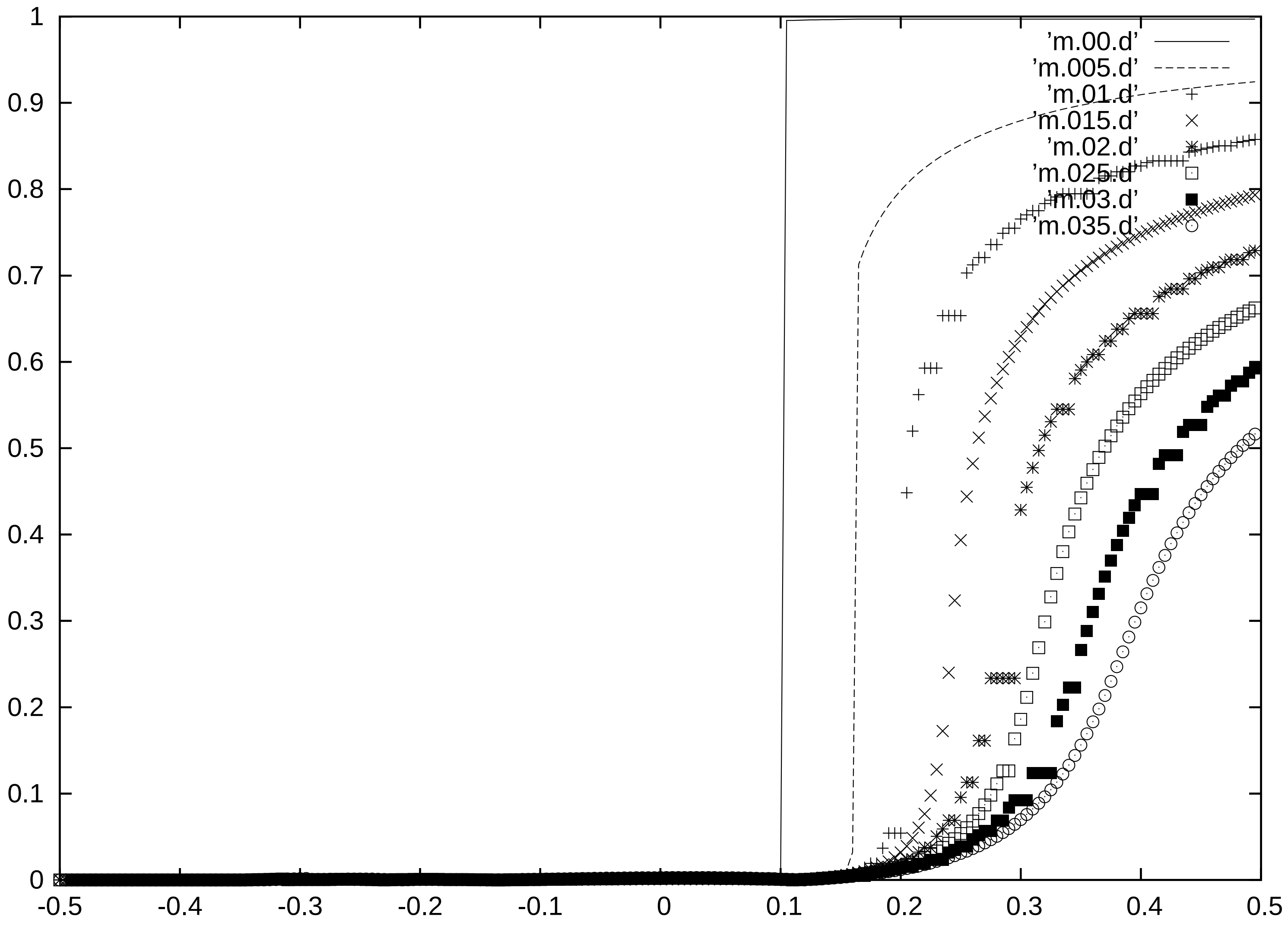
From

$$
\begin{aligned}
P_2(1,2) = {} & \frac{2}{3}\alpha(\Gamma)\mathbf{STr}_{\mathbf{3},\mathbf{4}} P_4(S_1,S_2,S_3,S_4) \\
& + \beta(\Gamma)\mathbf{STr}_{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}} P_6(S_1,S_2,S_3,S_4,S_5,S_6) \\
& + \frac{4}{3}\gamma(\Gamma)\mathbf{STr}_{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}} P_8(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8)
\end{aligned}
\tag{30}
$$

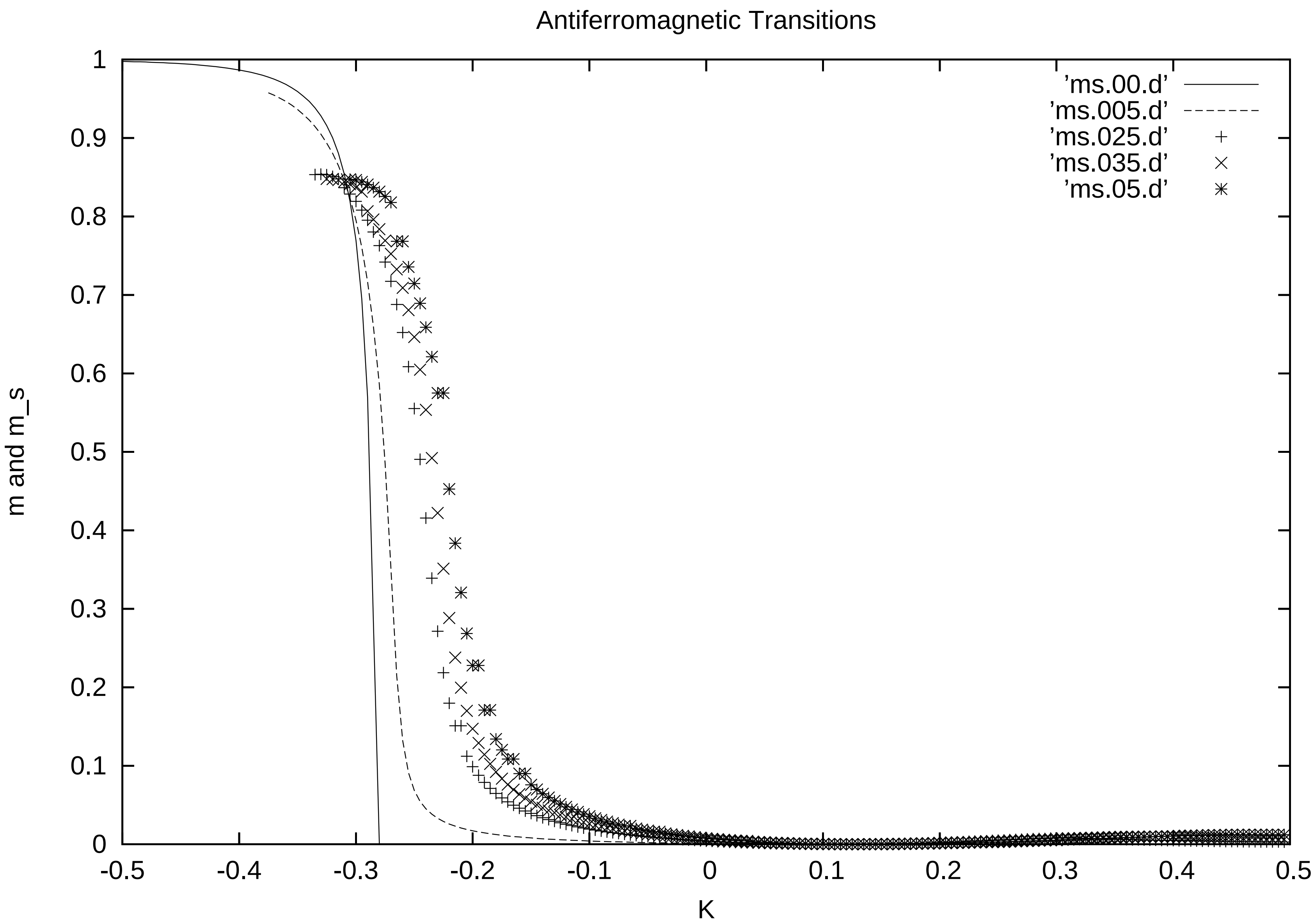
We obtain

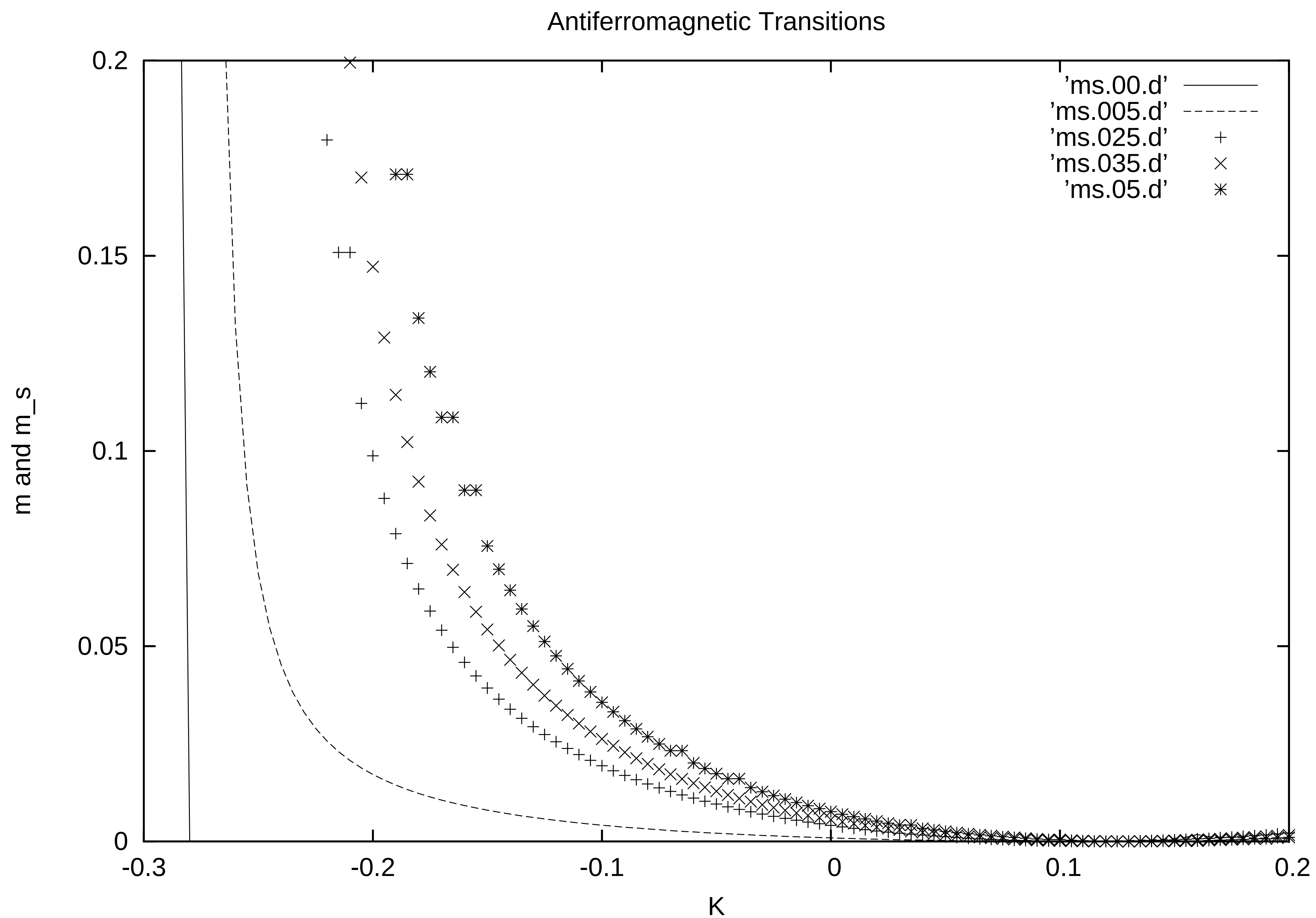
$$
x = \frac{1}{u}\frac{\frac{2}{3}\alpha\frac{1+x}{4+2x} + \beta\frac{x+4x^2+x^3}{2+6x+12x^2+2x^3} + \frac{4}{3}\alpha\frac{3+9x+8x^2+x^3}{24+36x+24x^2+2x^3}}{\frac{2}{3}\alpha\frac{1}{4+2x} + \beta\frac{1+2x+2x^2}{2+6x+12x^2+2x^3} + \frac{4}{3}\alpha\frac{9+9x+4x^2}{24+36x+24x^2+2x^3}} \tag{31}
$$

Ferromagnetic Phase Transitions

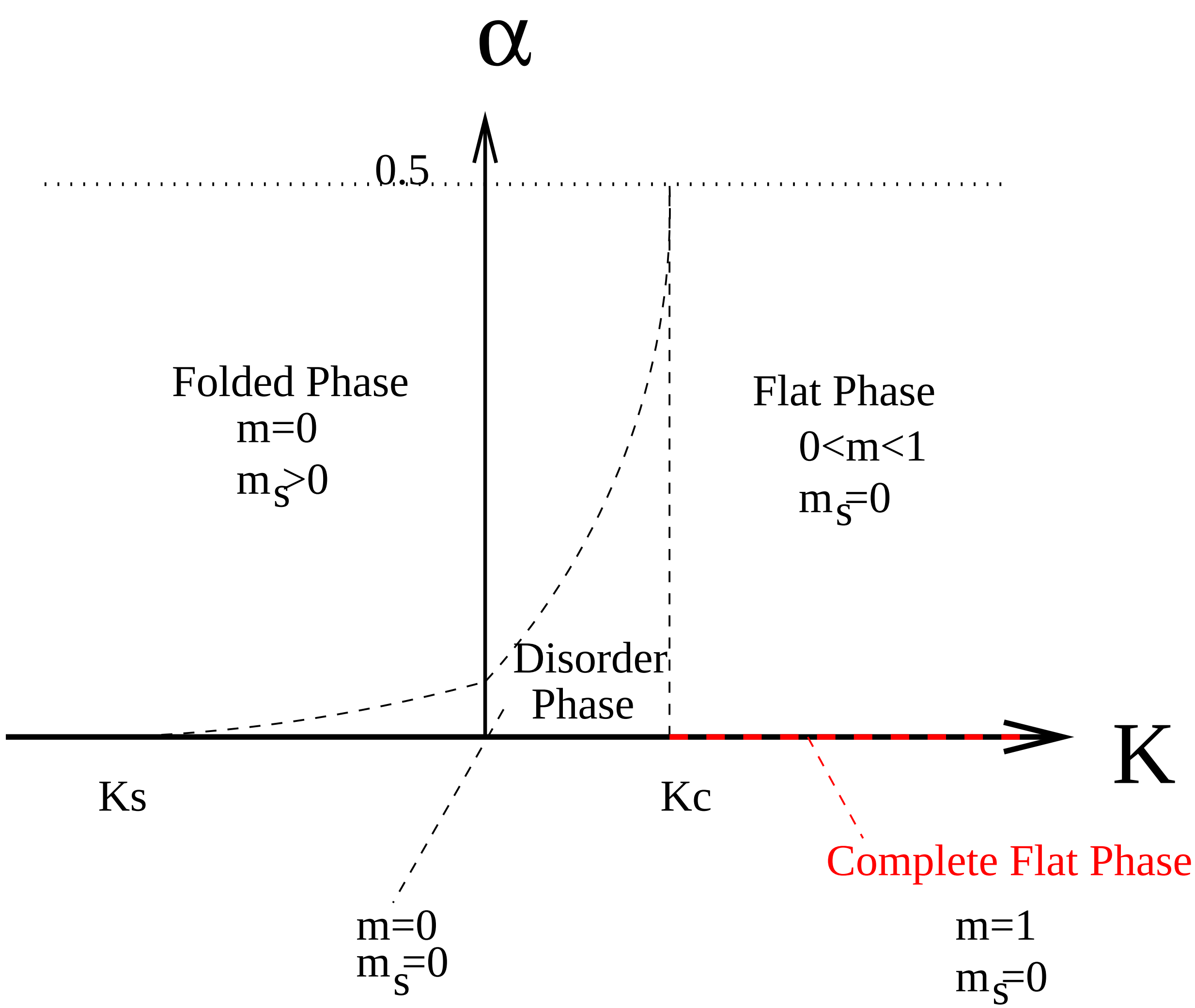


Antiferromagnetic Phase Transitions




Antiferromagnetic Transitions
'ms.00.d'
'ms.005.d'
'ms.025.d'
'ms.035.d'
'ms.05.d'
m and m_s
0.2
0.15
0.1
0.05
0
-0.3
-0.2
-0.1
0
0.1
0.2
K

Phase Diagram in (K, α)-Plane



2.1 Future Problem

- Generation of Γ and more elaborate Free Energy
  $P(\Gamma)$?

- Geometric Properties (Monte Carlo, Paralel-Tempering)
  Even Pure Case is non-Trivial

Today's Topic is in

Phase Transitions of the Randomly Triangulated Surface, S.Mori (in Preparation).