ランダム三角形分割された膜の統計物理 II

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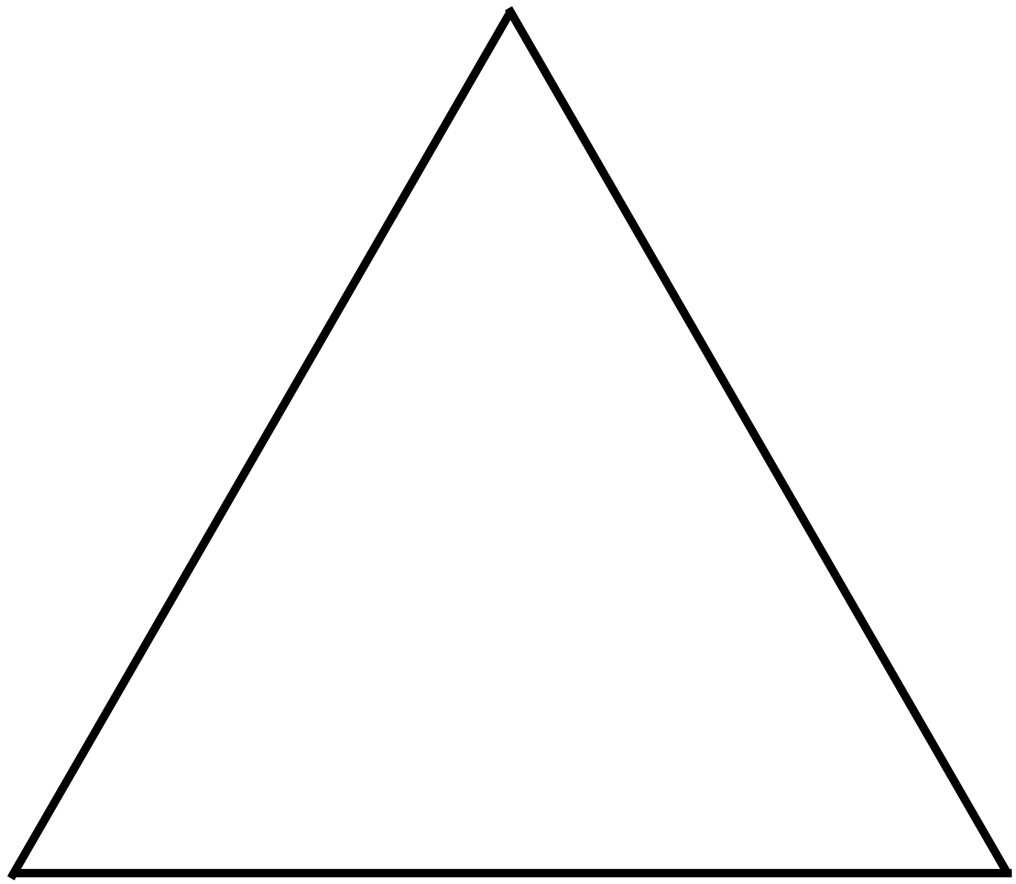
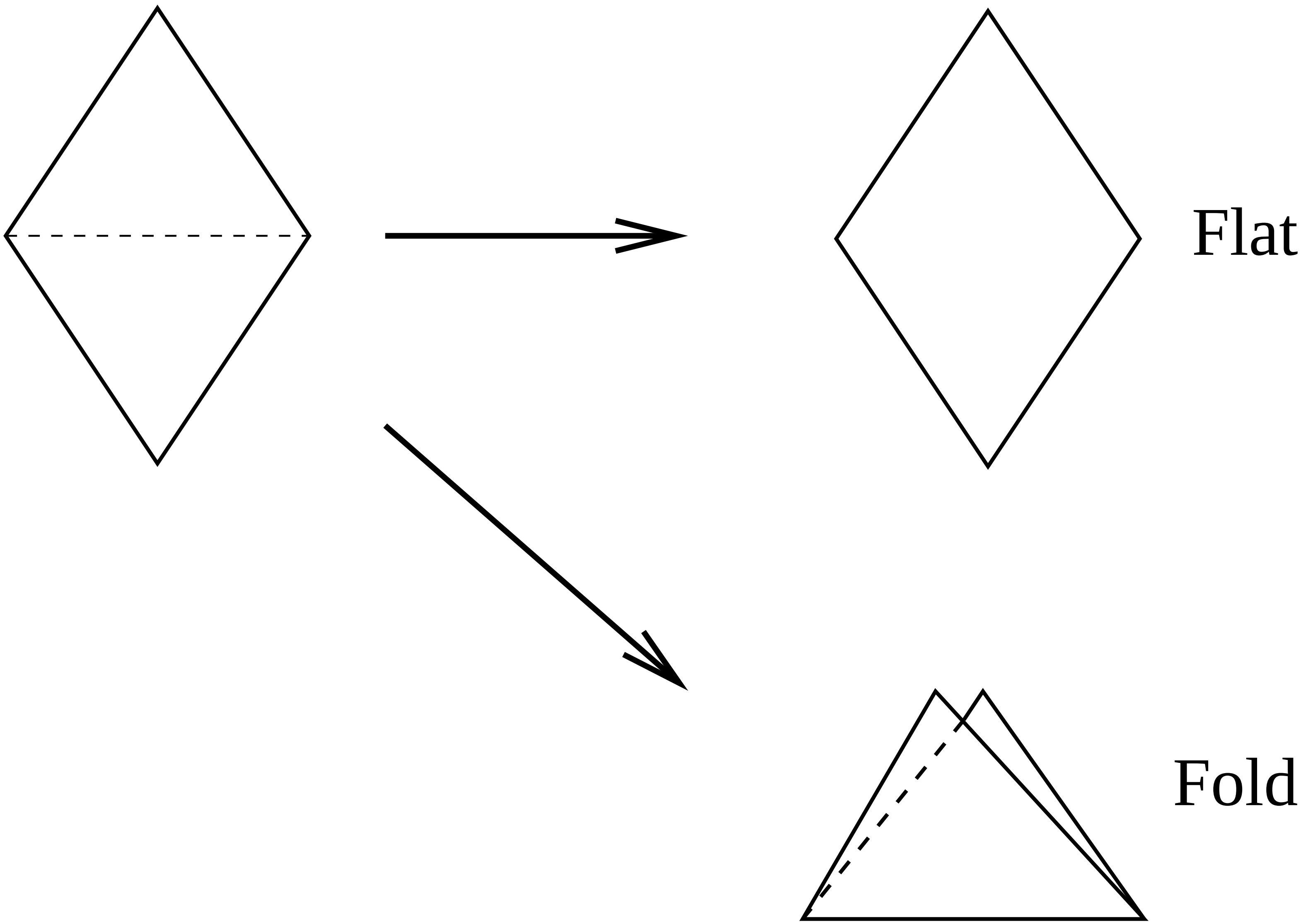
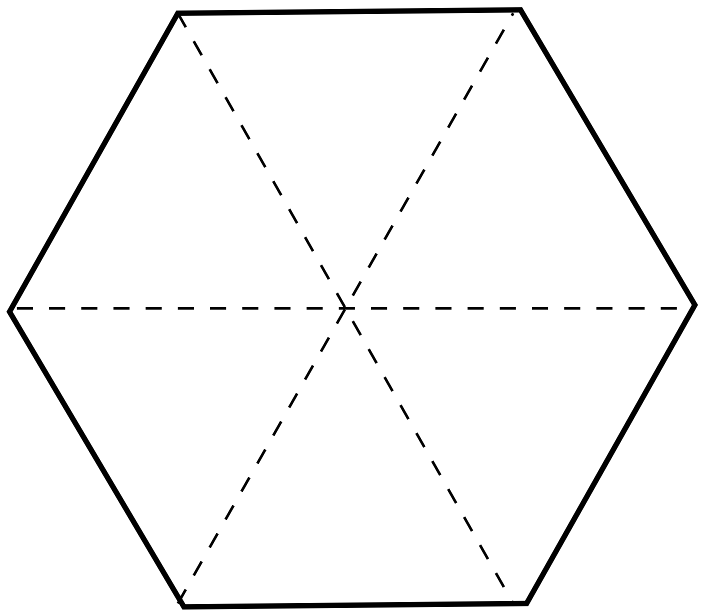
Contents

1　三角格子の折り畳みとは?



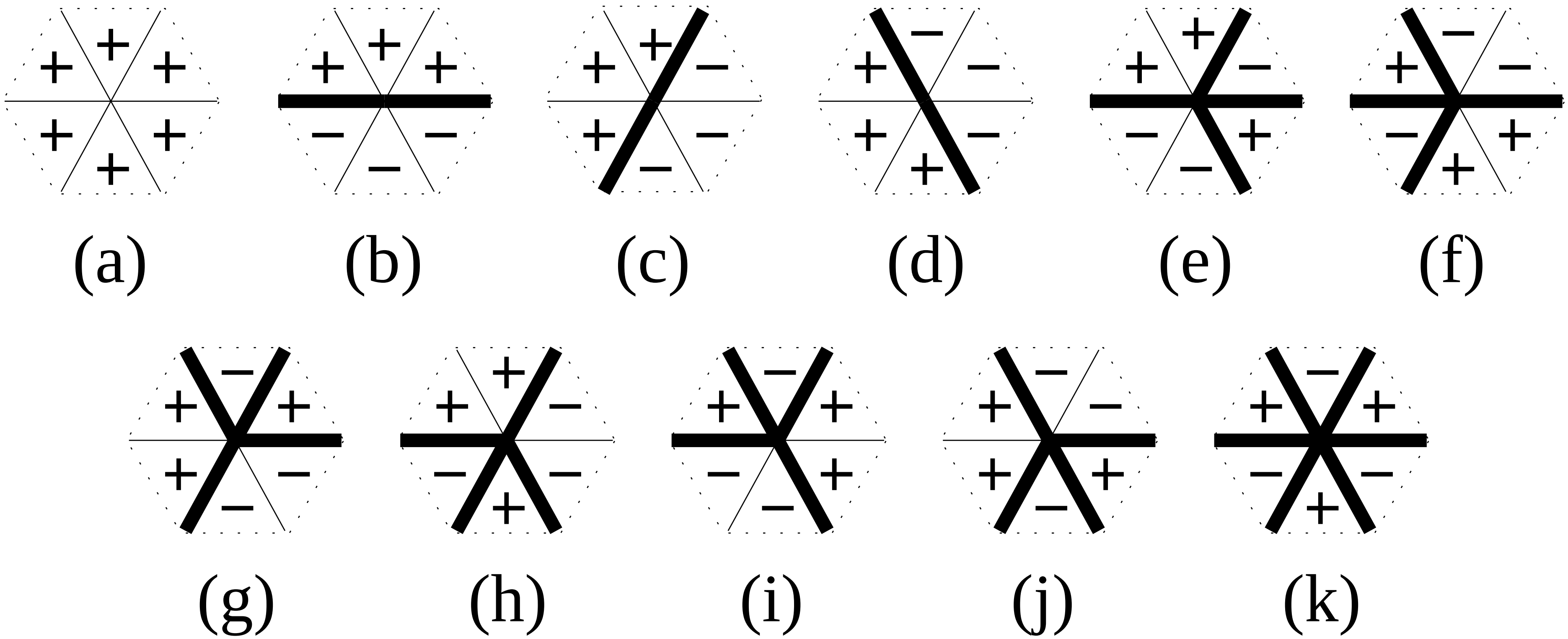

Flat
Fold

Elementary Hexagon



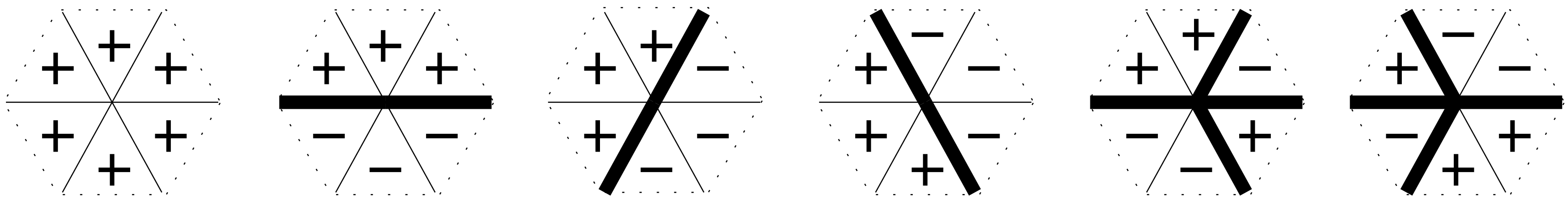
6 Bonds ----> 64 States ?

Only 11 Physical States

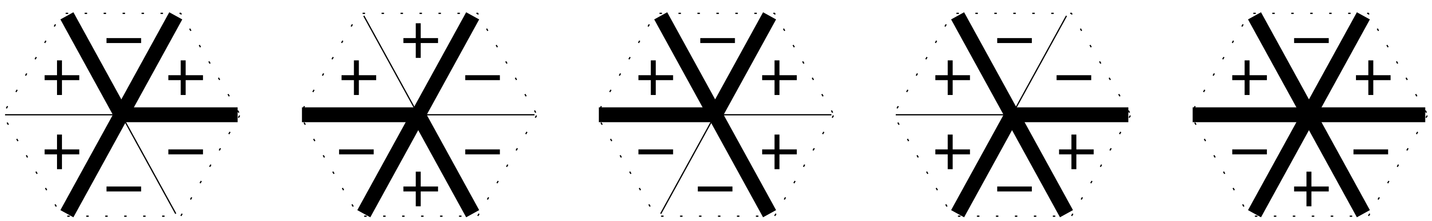

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)

Face Up $- - -- > S = +1$

Face Down $- - - > S = -1$



(a) (b) (c) (d) (e) (f)

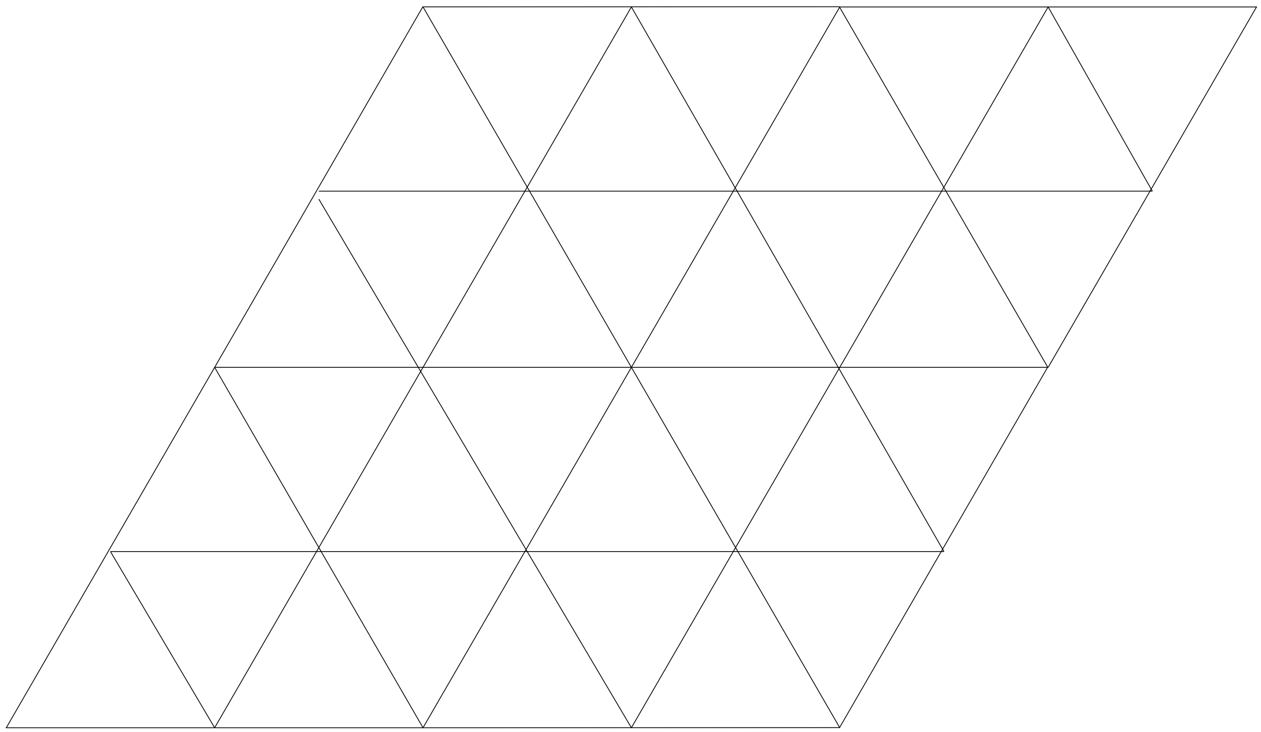


(g) (h) (i) (j) (k)

Folding Constraints (or Geometric Constraints)

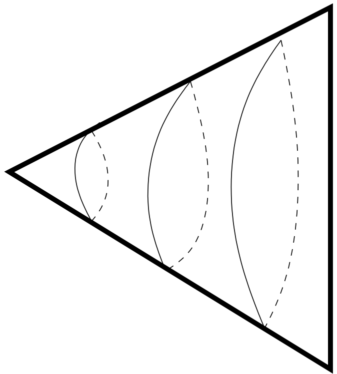
$$\sigma_v \equiv \sum_{i\ around\ v} S_i = 0 \text{ mod } 3 \tag{1}$$

Regular Triangular Lattice



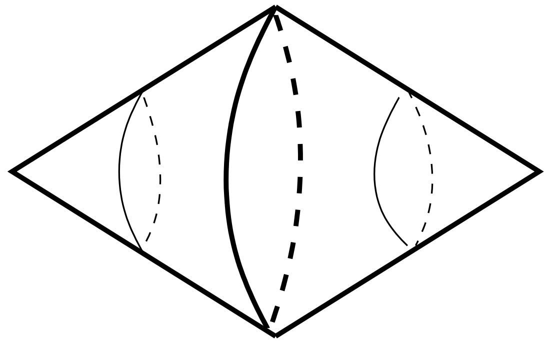
Eulerian Randomly Triangulated Surface

2 Faces



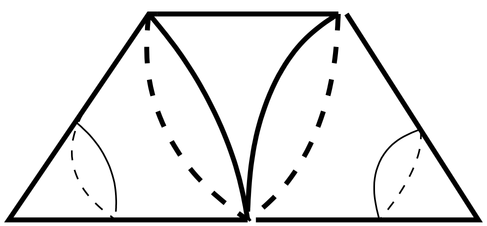
1

4 Faces

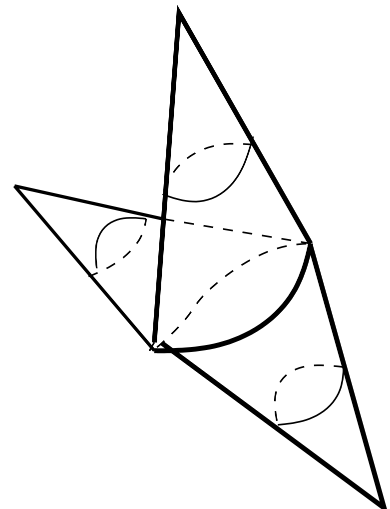


3

6 Faces



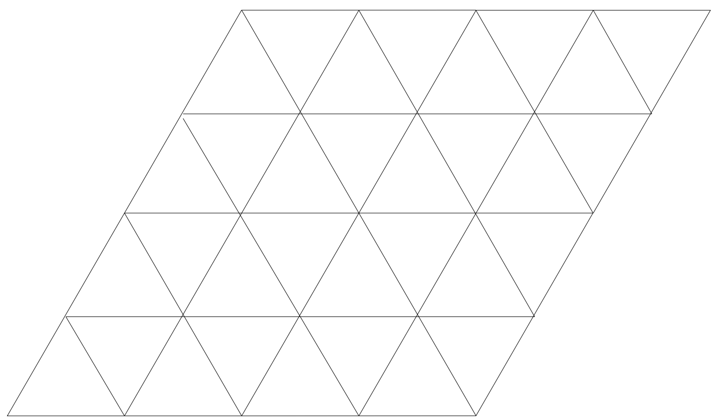
9



3

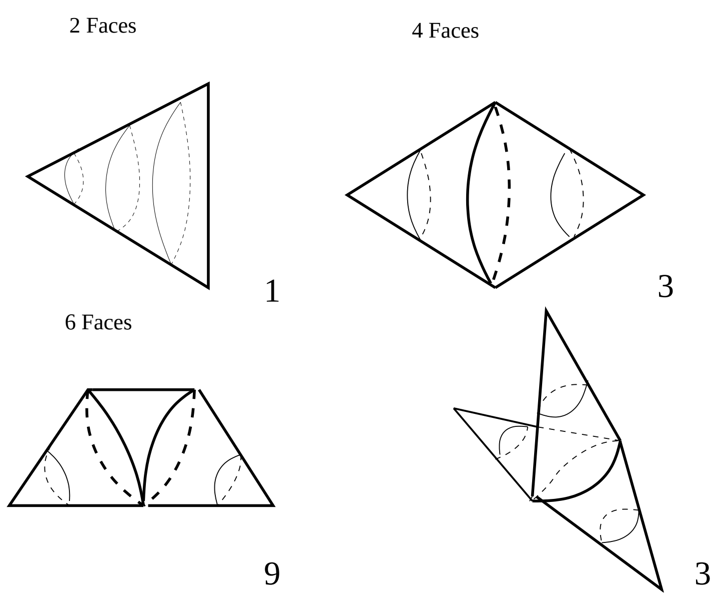
Toy Model for Polymer(ized) Membrane:

- Folding of Triangular Lattice
- Constrained Z_2 Spin System on the Dual of Triangular Lattice
- 3-Coloring Problem of Triangular Lattice



Toy Model for Fluid Membrane:

- Folding of Eulerian Randomly Triangulated Surface
- Constrained Z_2 Spin System on its Dual Random Diagram
- 3-Coloring Problem of the Random Diagram



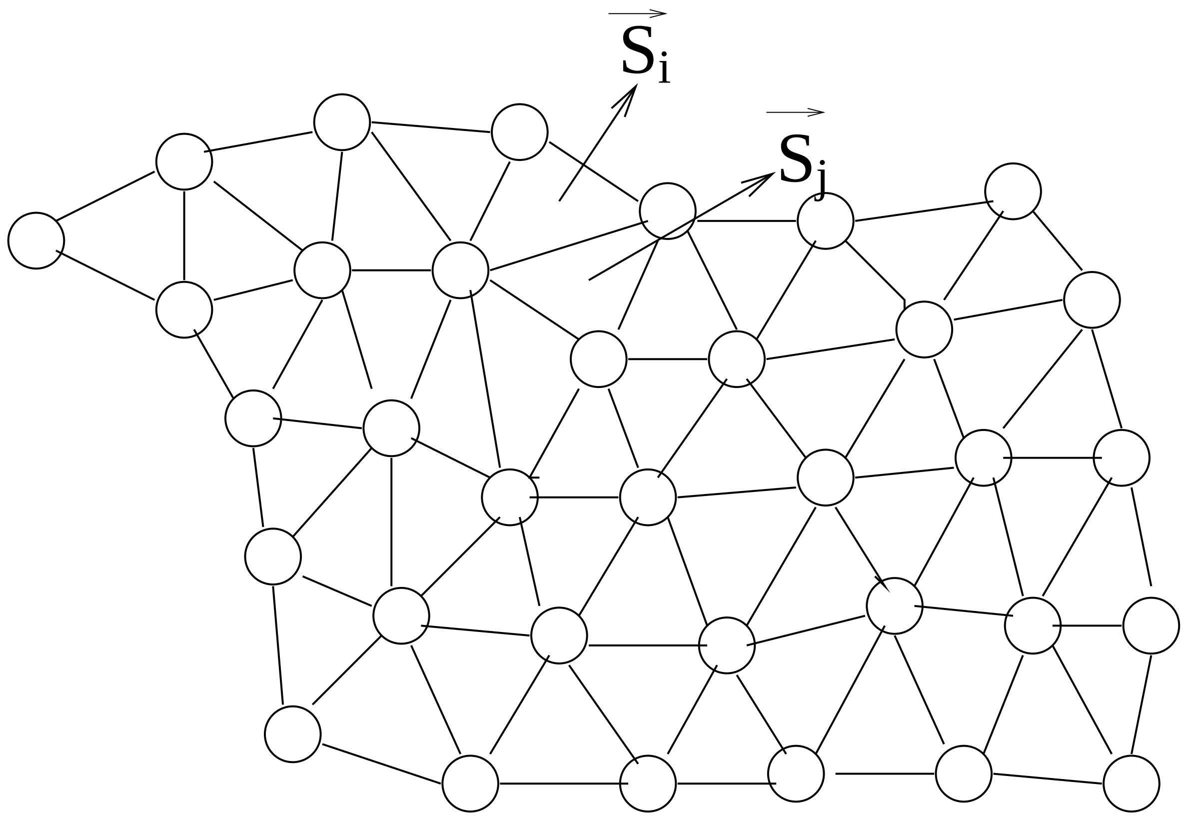
How to Formulate.

- Matrix Model ?

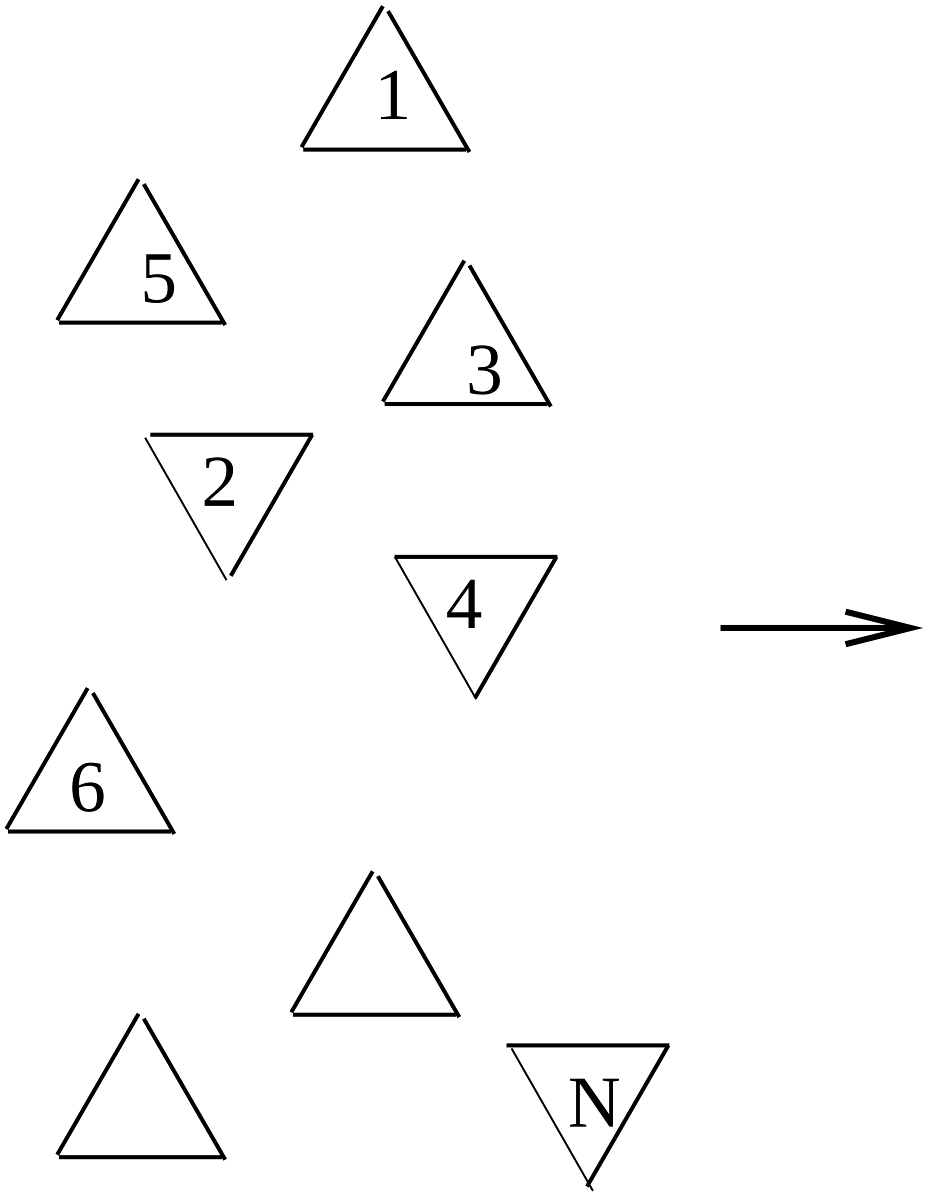
  Difficult and not yet solved
- Decorated Tree

  also difficult and under progress
- Mean Field (Cluster Variation) Approach

2　ランダム三角形分割された膜の折り畳み問題と相転移



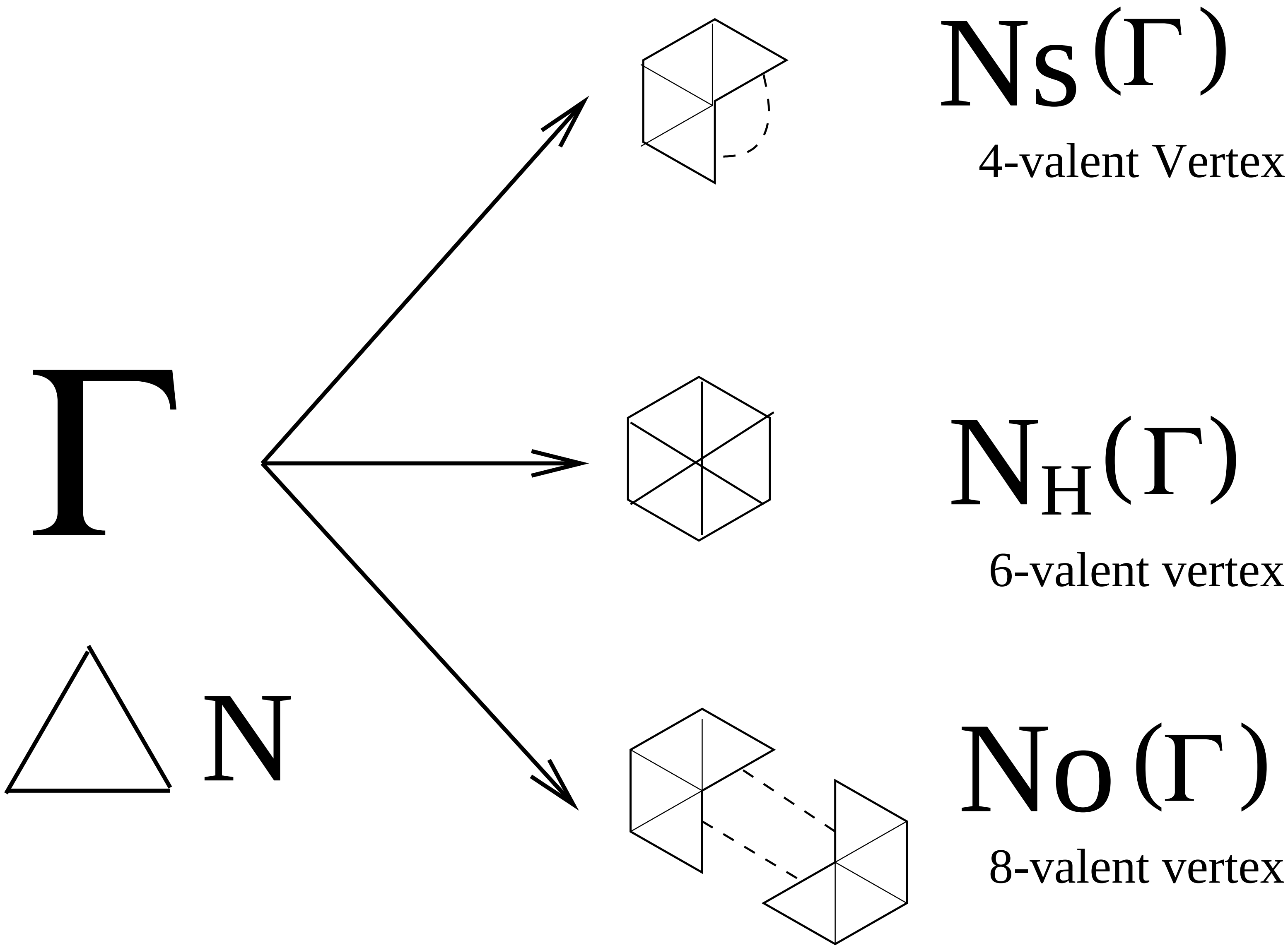
$$-\beta\mathcal{H} = K \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \tag{2}$$



Eulerian Random

Triangulated Lattice

$$\Gamma$$


Ns(Γ)
4-valent Vertex
Γ
N_H(Γ)
6-valent vertex
N
No(Γ)
8-valent vertex

Folding of $\Gamma \quad \rightarrow \quad i = S_1, S_2, \cdots, S_N$

with the Geometric Constraints

$$\sigma_v \equiv \sum_{i \ around \ v} S_i = 0 \text{ mod } 3$$

Hamiltonian

$$-\beta\mathcal{H}(i|\Gamma) = K \sum_{<ij>} S_i S_j + h \sum_i S_i \tag{3}$$

$$K = \beta J = J/k_B T \quad \text{and} \quad h = \beta H \tag{4}$$

N_B : Number of Spin-Pairs

$$N_B = \frac{3}{2}N$$

From Euler's Relation,

$$N_S + N_H + N_O \simeq \frac{1}{2}N = N_F$$

and the Relation

$$\frac{1}{2}(4 \times N_S + 6 \times N_H + 8 \times N_O) = N_B$$

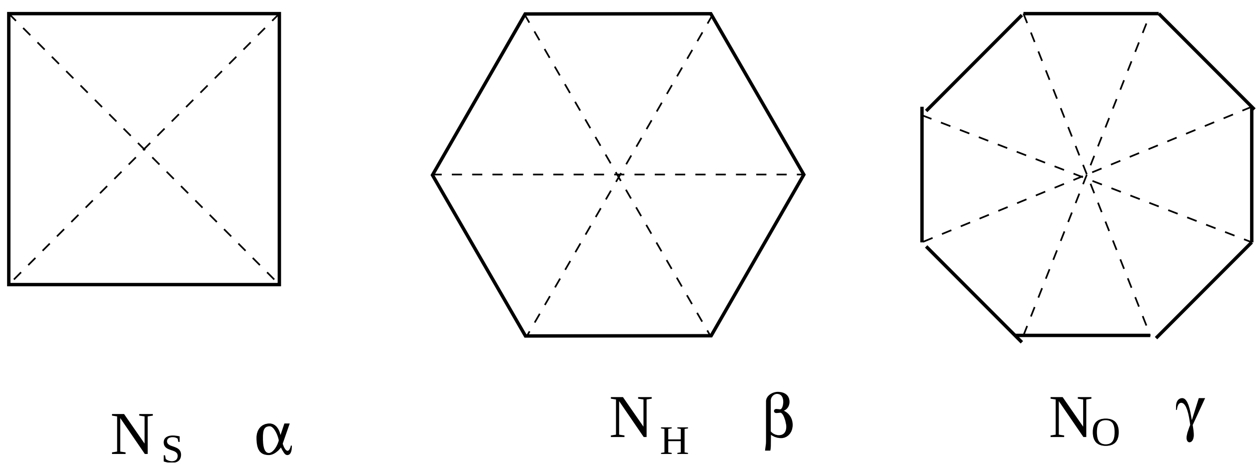
We obtain

$$N_S = N_O$$

Parameters α, β, γ:

$$\alpha(\Gamma) = \frac{N_S(\Gamma)}{N_F} \quad \beta(\Gamma) = \frac{N_H(\Gamma)}{N_F} \quad \gamma(\Gamma) = \frac{N_O(\Gamma)}{N_F}$$

$$\alpha(\Gamma) = \gamma(\Gamma) \text{ and } \beta(\Gamma) = 1 - 2\alpha(\Gamma)$$



$$
\begin{aligned}
f[\Gamma] \equiv\ & \frac{1}{N} F[\Gamma] \\
=\ & -\frac{3}{2} K \mathbf{Tr}_{\mathbf{1},\mathbf{2}} P_2(S_1, S_2) S_1 S_2 - \frac{1}{2} h \mathbf{Tr}_{\mathbf{1}} P_{1A}(S_1) S_1 - \frac{1}{2} h \mathbf{Tr}_{\mathbf{2}} P \\
& -\frac{1}{N} S_{spin}[\Gamma] \\
& +\lambda_8 (\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8}} P_8(S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8) - 1) \\
& +\lambda_6 (\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}} P_6(S_1, S_2, S_3, S_4, S_5, S_6) - 1) \\
& +\lambda_4 (\mathbf{Tr}_{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}} P_4(S_1, S_2, S_3, S_4) - 1)
\end{aligned}
$$

Natural Iteration Method (or Variational Equation)

$$P_N(S_1, S_2, \cdots, S_N) = \exp(-\lambda_N + \frac{K}{2}\sum_{i=1}^{N} S_i S_{i+1} + \frac{h}{3}\sum_{i=1}^{N} S_i)$$

$$\times\ (P_2(S_1, S_N)P_2(S_1, S_2)P_2(S_3, S_2)\cdots P_2(S_{N-1}, S_N))$$

$$\times\ (P_{1A}(S_1)P_{1B}(S_2)\cdots P_{1A}(S_{N-1})P_{1B}(S_N))^{-\frac{1}{3}} \qquad \textbf{(6}$$

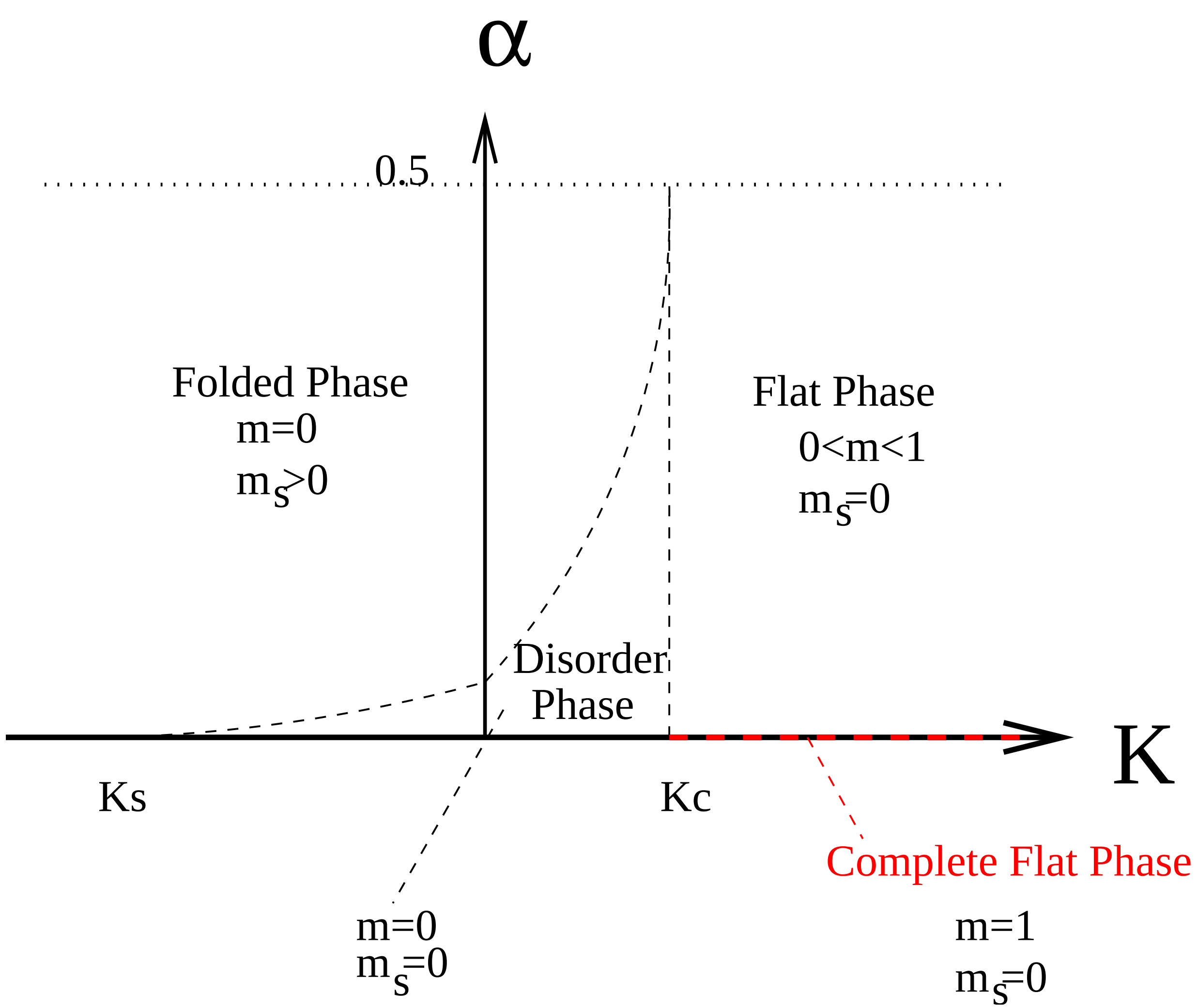
with

$$S_{N+1} = S_1$$

and

$$N = 4, 6, 8$$

Phase Diagram in (K, α)-Plane



2.1 Future Problem

- Generation of Γ and more elaborate Free Energy
- Geometric Properties (Monte Carlo, Paralel-Tempering)
  Even Pure Case is non-Trivial.
- Embedded in F.C.C Lattice Case.
  Geometric Folding Transition ?

Today's Topic is in

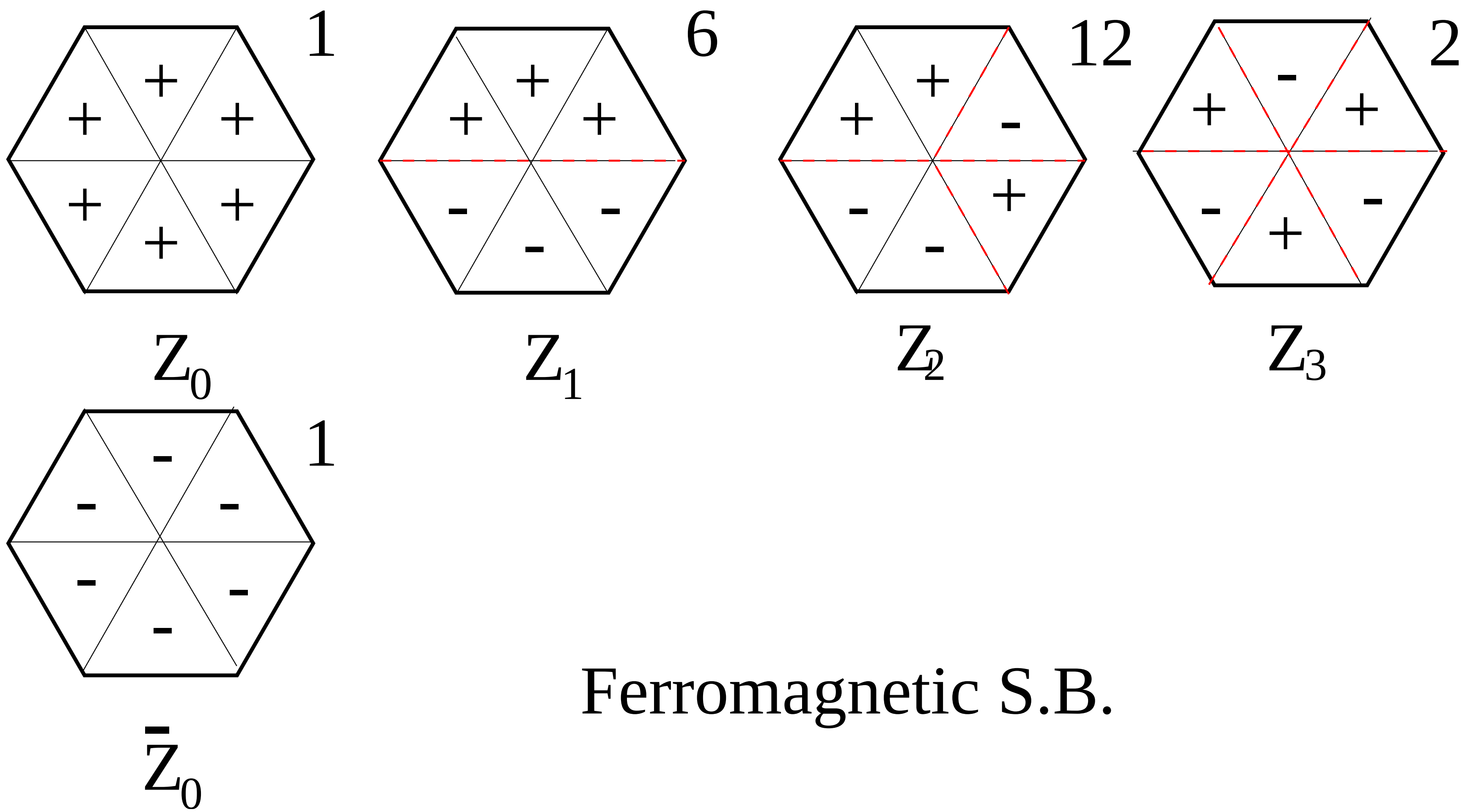
Phase Transitions of the Randomly Triangulated Surface, S.Mori (in Preparation).

Regular Triangular Lattice Case ($\beta = 1$)

Config.	M	M_S	deg.	Ferro. S.B.	Antiferro S.B.
$++++++$	6	0	1	Z_0	Z_0
$------$	-6	0	1	$\overline{Z_0}$	Z_0
$+++---$	0	2	3	Z_1	Z_1
$---+++$	0	-2	3	Z_1	$\overline{Z_1}$
$+--++-$	0	2	6	Z_2	Z_2
$-++--+$	0	-2	6	Z_2	$\overline{Z_2}$
$+-+-+-$	0	6	1	Z_3	Z_3
$-+-+-+$	0	-6	1	Z_3	$\overline{Z_3}$

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$



Ferromagnetic S.B.

Ferromagnetic S.B. Case :

Introducing x, y as

$$
\begin{aligned}
x &= \frac{Z_1}{\sqrt{Z_0 \overline{Z_0}}} \equiv \frac{Z_1}{W} \qquad \text{Fugacity per Fold} \\
y &= \frac{Z_0}{\overline{Z_0}} \qquad \text{Order Parameter}
\end{aligned}
\tag{7}
$$

$$
Z_0 = W y^{\frac{1}{2}} \qquad \overline{Z_0} = W y^{-\frac{1}{2}} \qquad Z_1 = W x \qquad Z_2 = W x^2 \qquad Z_3 = W x^3
$$

Algebraic Relation between x and y as

$$y = \left(\frac{y^{\frac{1}{2}} + 2x + 2x^2}{y^{\frac{-1}{2}} + 2x + 2x^2}\right)^3 \left(\frac{y^{\frac{-1}{2}} + 3x + 6x^2 + x^3}{y^{\frac{1}{2}} + 3x + 6x^2 + x^3}\right)^2 \tag{8}$$

$$x = e^{-2K} \frac{\left(x + 4x^2 + x^3\right)}{\left(\left(y^{\frac{1}{2}} + 2x + 2x^2\right)\left(y^{\frac{-1}{2}} + 2x + 2x^2\right)\right)^{\frac{1}{2}}} \tag{9}$$

Solutions:

- $y = 1$: Disordered Phase

$$x = \frac{2 - u + \sqrt{3 - u - u^2}}{2u - 1} \quad \text{with} \quad u = e^{2K}$$

$$x = 2 \qquad c =< S_1 S_2 >= -\frac{1}{3} \quad \text{at} \quad K = 0$$

- $x = 0$ and $y =$arbitrary : Completely Flat Phase

Anti-Ferromagnetic S.B. Case :

Introducing x, y as

$$x = \frac{\sqrt{Z_1 \overline{Z_1}}}{Z_0} \quad \text{Fugacity per Fold}$$
$$y = \frac{Z_1}{\overline{Z_1}} \quad \text{Order Parameter} \tag{10}$$

$$Z_1 = W y^{\frac{1}{2}} x \qquad \overline{Z_1} = W y^{-\frac{1}{2}}$$
$$Z_2 = W y^{\frac{1}{2}} x^2 \qquad \overline{Z_2} = W y^{-\frac{1}{2}} x^2$$
$$Z_3 = W y^{\frac{3}{2}} x^3 \qquad \overline{Z_3} = W y^{-\frac{3}{2}} x^3 \tag{11}$$

Algebraic Relation between x and y as

$$y = \frac{xy^{\frac{1}{2}} + 3x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}} + x^3y^{\frac{3}{2}}}{xy^{-\frac{1}{2}} + 3x^2y^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^3y^{-\frac{3}{2}}} \left(\frac{1 + 2xy^{-\frac{1}{2}} + xy^{\frac{1}{2}} + 2x^2y^{\frac{1}{2}} + 4x^2y^{-\frac{1}{2}} + x^3y^{-\frac{3}{2}}}{1 + 2xy^{\frac{1}{2}} + xy^{-\frac{1}{2}} + 2x^2y^{-\frac{1}{2}} + 4x^2y^{\frac{1}{2}} + x^3y^{\frac{3}{2}}}\right)^{\frac{2}{3}} \tag{12}$$

$$x = e^{-2K} \frac{(xy^{\frac{1}{2}} + 3x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}} + x^3y^{\frac{3}{2}})(xy^{-\frac{1}{2}} + 3x^2y^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^3y^{-\frac{3}{2}})}{1 + xy^{\frac{1}{2}} + xy^{-\frac{1}{2}} + x^2y^{\frac{1}{2}} + x^2y^{-\frac{1}{2}}} \tag{13}$$

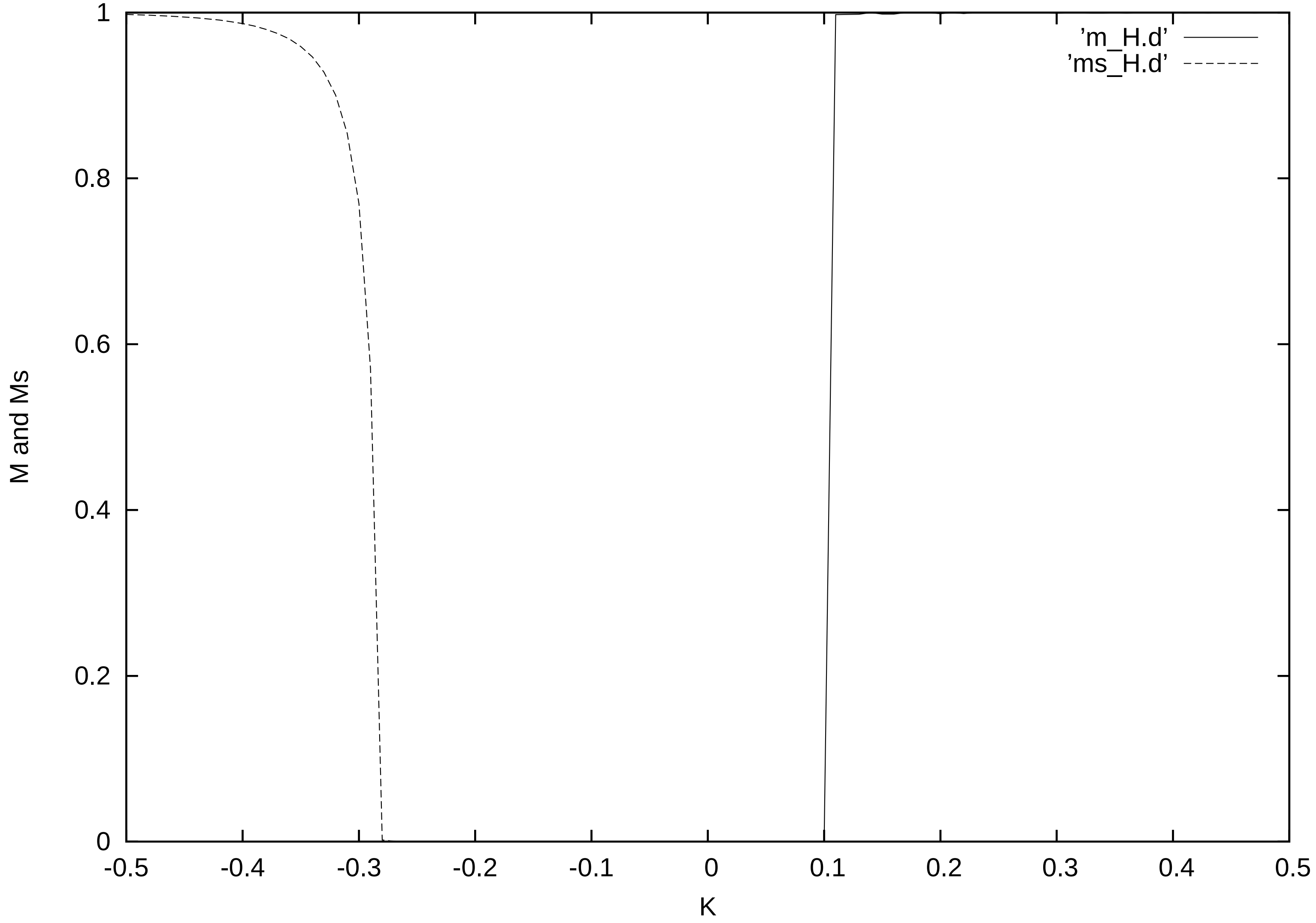
Solutions:

- $y = 1$: Disordered Phase
- $y = 1 + \epsilon$: Folded (Antiferromagnetic) Phase

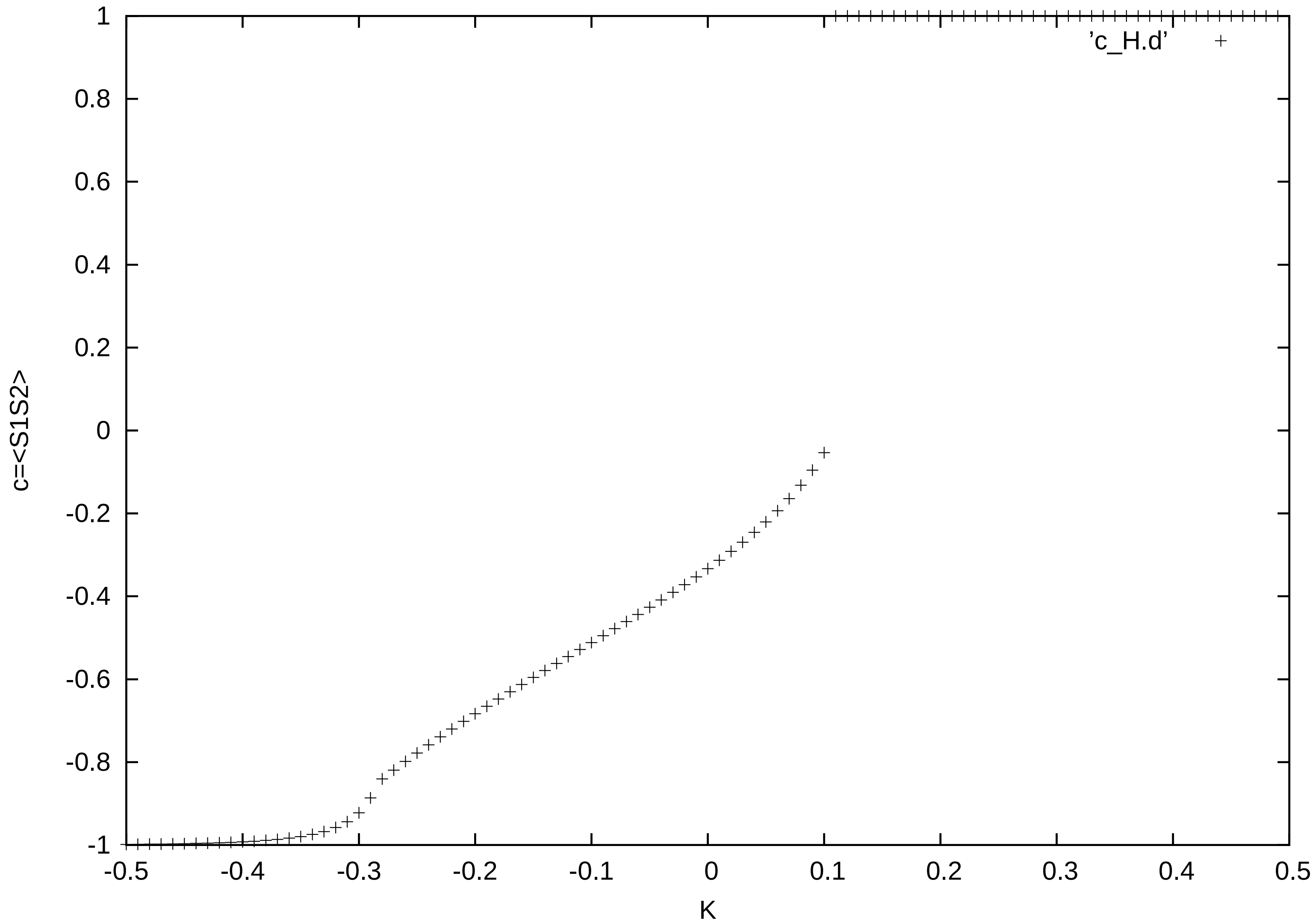
$$x_{st}^3 - 21x_{st}^2 - 12x_{st} - 4 = 0$$

$$K_{st} = -0.2838\cdots$$

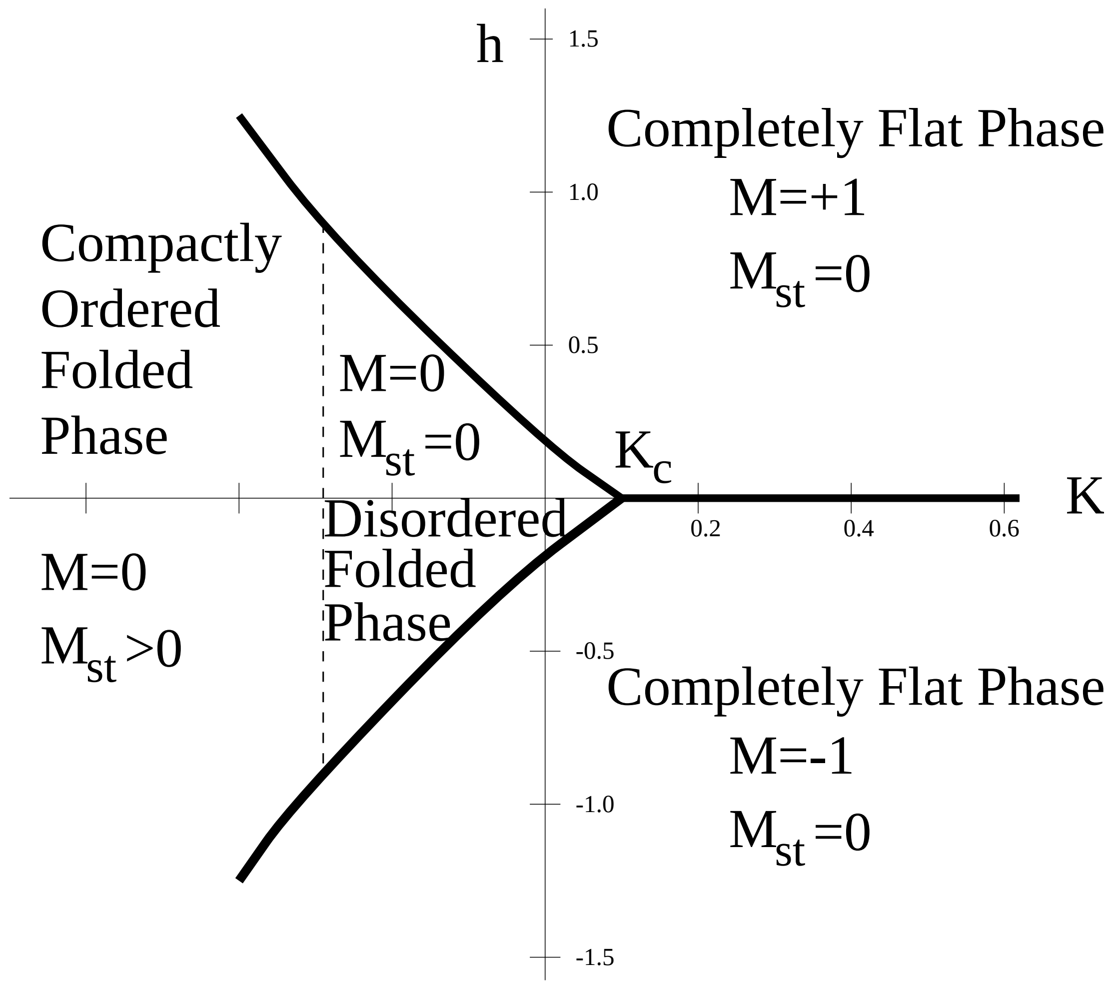
Phase Transitions



$c = \langle S_1 S_2 \rangle$



Phase Diagram:

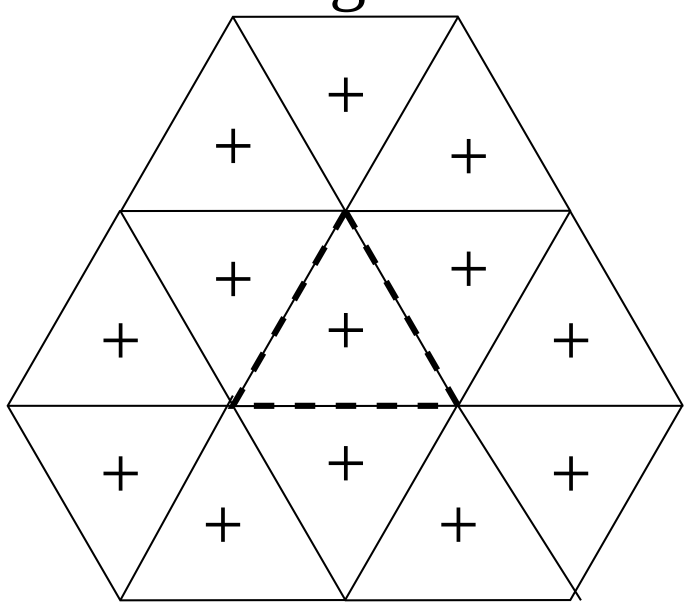


M.Cirillo, G.Gonella and A.Pelizzola ('95)

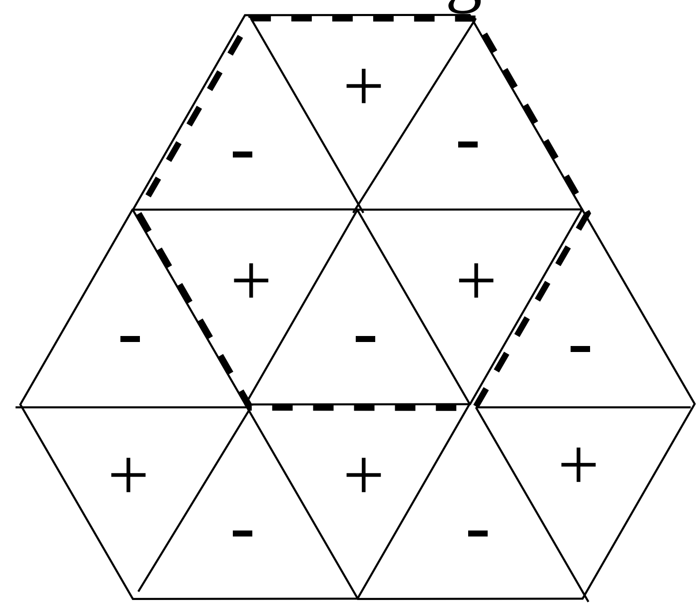
Ph. Di Francesco, E.Guitter and S.Mori ('96)

Local Excitation

Flat Config.



Piled Config.



Local Spin Flip

\+ ---> −

− ---> +

Impossible

Possible

Exact Results

Entropy or Number of States N_S for N Triangles:

$$N_S \sim q^N > 2^{\frac{1}{6}\times N}$$

$$q = \frac{\sqrt{3}}{2\pi}\Gamma(1/3)^{\frac{3}{2}} = 1.208717\cdots$$

Ph. Di Francesco and E.Guitter ('94)

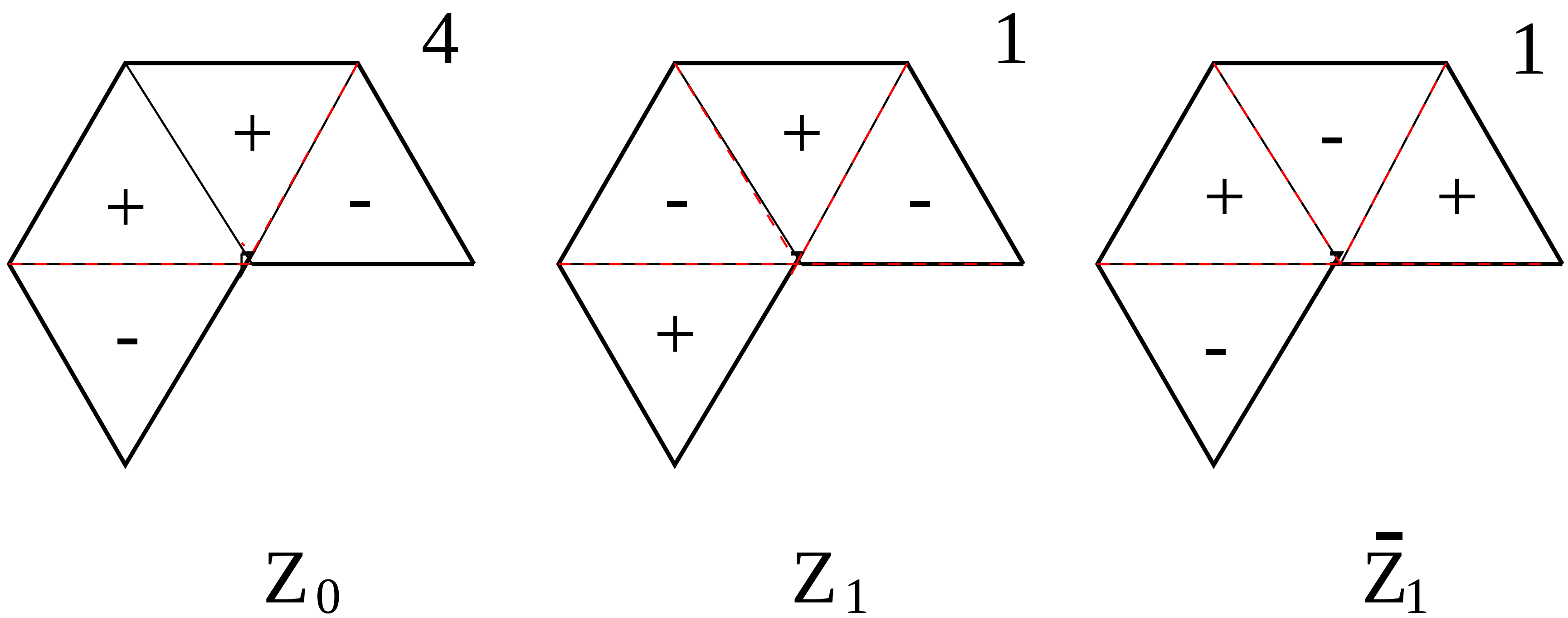
$$q_{CVM} = 1.2019$$

Pure 4-valent Veretx Case ($\alpha = 1$)

Confi g.	M	M_S	deg.	Antiferro S.B.
$++--$	0	0	4	Z_0
$+-+-$	0	4	1	Z_1
$-+-+$	0	-4	1	$\overline{Z_1}$

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$



AntiFerromagnetci S.B.

Introducing x, y as

$$x = \frac{\sqrt{Z_1 \overline{Z_1}}}{Z_0} \equiv \frac{Z_1}{W} \quad \text{Fugacity per Fold}$$
$$y = \frac{Z_1}{\overline{Z_1}} \quad \text{Order Parameter} \tag{14}$$

$$Z_1 = Wxy^{\frac{1}{2}} \qquad \overline{Z_1} = Wxy^{-\frac{1}{2}}$$

$$y = \frac{(1 + xy^{\frac{1}{2}})^2 (2 + xy^{-\frac{1}{2}})^{\frac{4}{3}}}{(1 + xy^{-\frac{1}{2}})^2 (2 + xy^{\frac{1}{2}})^{\frac{4}{3}}} \tag{15}$$
$$x = e^{-2K}(1 + xy^{\frac{1}{2}})(1 + xy^{-\frac{1}{2}}) \tag{16}$$

Solutions:

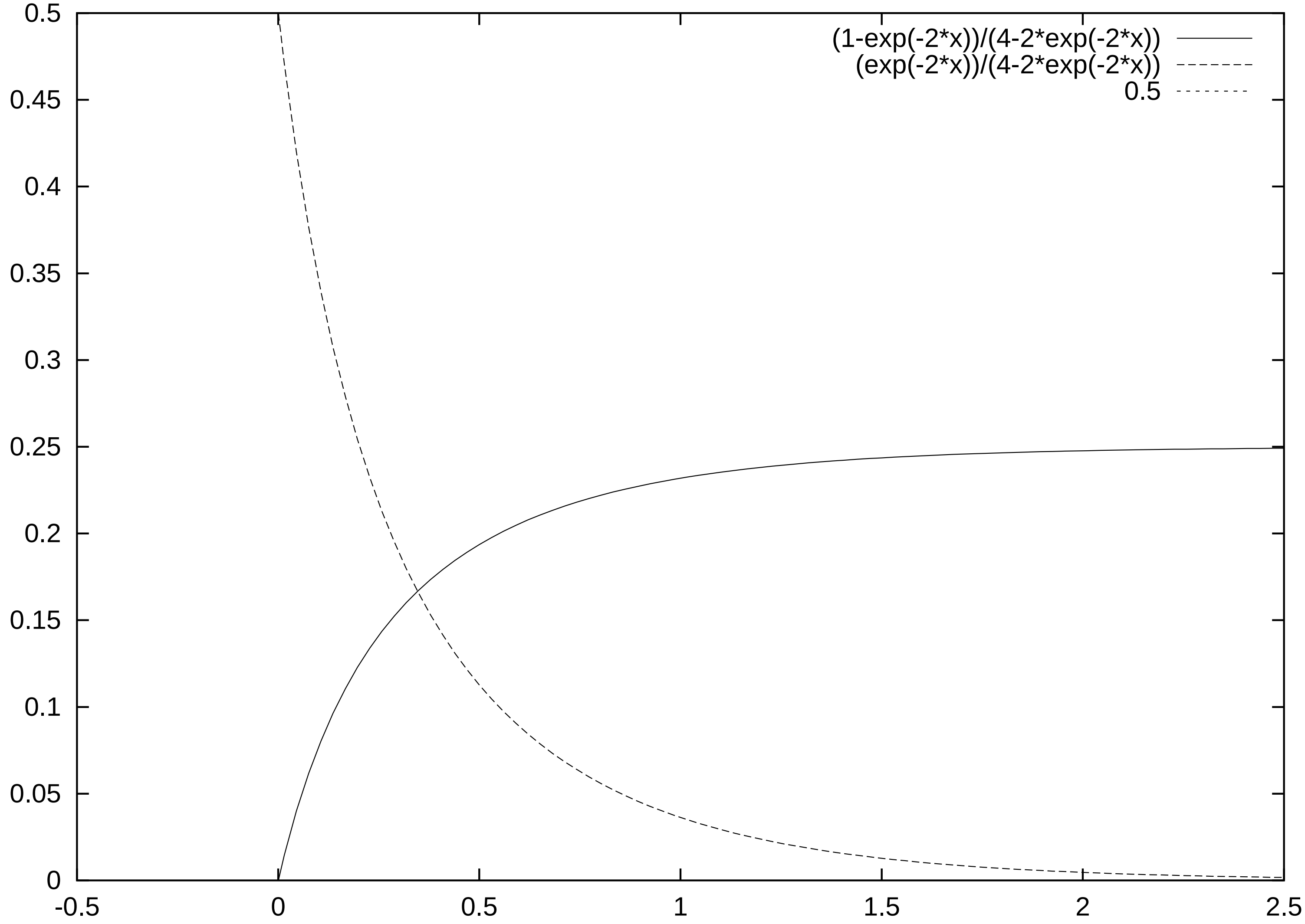
- $y = 1$: Disordered Phase

$$
\begin{aligned}
x &= \frac{e^{-2K}}{1 - e^{-2K}} \\
Z_0 &= \frac{1 - e^{-2K}}{4 - 2e^{-2K}} \\
Z_1 &= \overline{Z_1} = \frac{e^{-2K}}{4 - 2e^{-2K}} \\
c &= < S_1 S_2 > = -2 \times Z_1
\end{aligned}
\tag{17}
$$

- $y = 1 + \epsilon$ does not have Real Solution for K

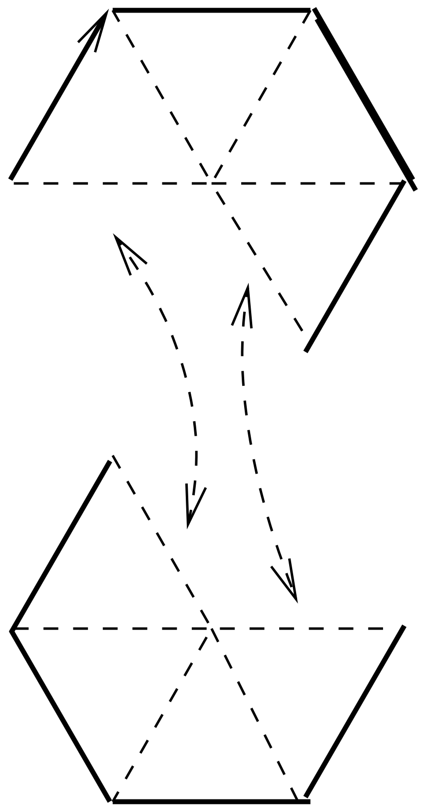
No Folded (Antiferromagnetic) Phase. Always Disorderd.

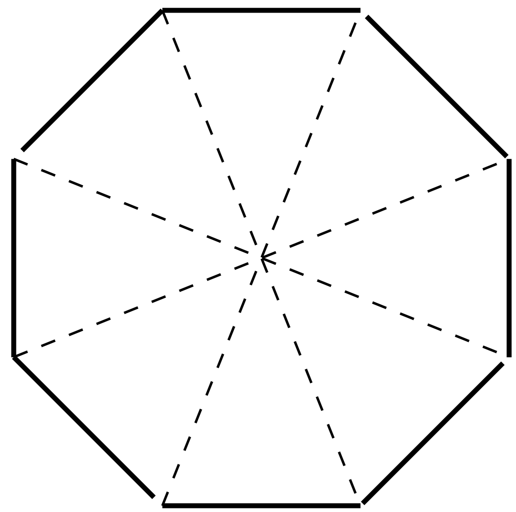
Z_0 and Z_1


(1-exp(-2*x))/(4-2*exp(-2*x))
(exp(-2*x))/(4-2*exp(-2*x))
0.5
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.5
0
0.5
1
1.5
2
2.5

Pure 8-Valent Vertex Case ($\gamma = 1$)

Foldable





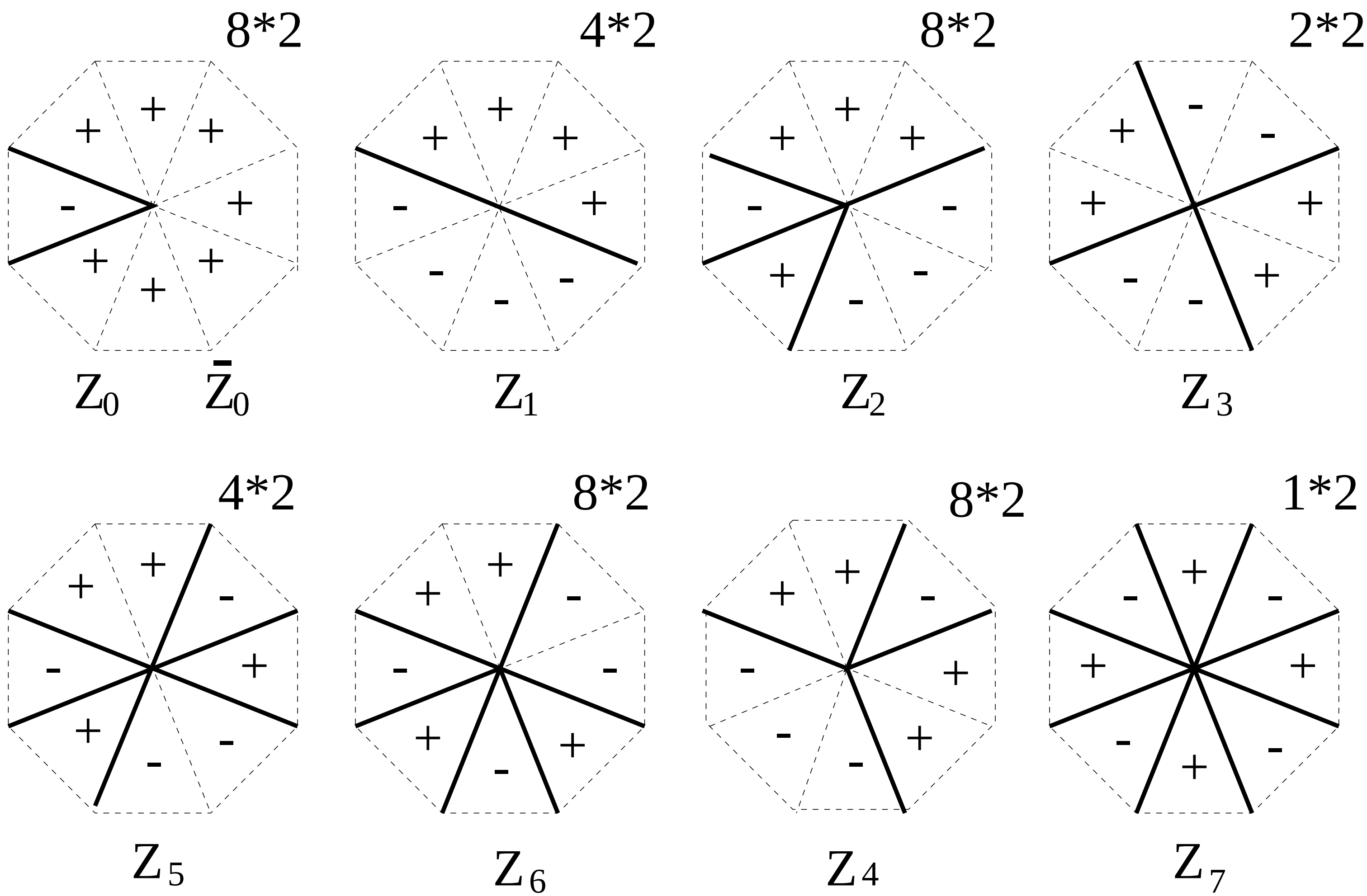
8 Bonds

43 States

Confi g.	M	deg.	Ferro. S.B.
$+ + + + + + + -$	6	8	Z_0
$+ - - - - - - -$	-6	8	$\overline{Z_0}$
$+ + + + - - - -$	0	8	Z_1
$+ + + - - - + -$	0	16	Z_2
$+ + - - + + - -$	0	4	Z_3
$+ - - + + + - -$	0	16	Z_4
$+ - + + - + - -$	0	8	Z_5
$+ + - + - + - -$	0	16	Z_6
$+ - + - + - + -$	0	2	Z_7

$M = \sum_i S_i$

$M_S = \sum_i (-1)^{i-1} S_i$



Ferromagnetic S.B.

Ferromagnetic S.B. Case :

Introducing x, y as

$$x = \frac{Z_2}{\sqrt{Z_0 \overline{Z_0}}} \equiv \frac{Z_2}{W} \quad \text{Fugacity per Fold}$$

$$y = \frac{Z_0}{\overline{Z_0}} \quad \text{Order Parameter} \tag{18}$$

$$Z_0 = W y^{\frac{1}{2}} \quad \overline{Z_0} = W y^{-\frac{1}{2}}$$
$$Z_1 = W \quad Z_2 = W x$$
$$Z_3 = W x \quad Z_4 = W x$$
$$Z_5 = W x^2 \quad Z_6 = W x^2$$
$$Z_7 = W x^3 \tag{19}$$

Algebraic Relations between x and y

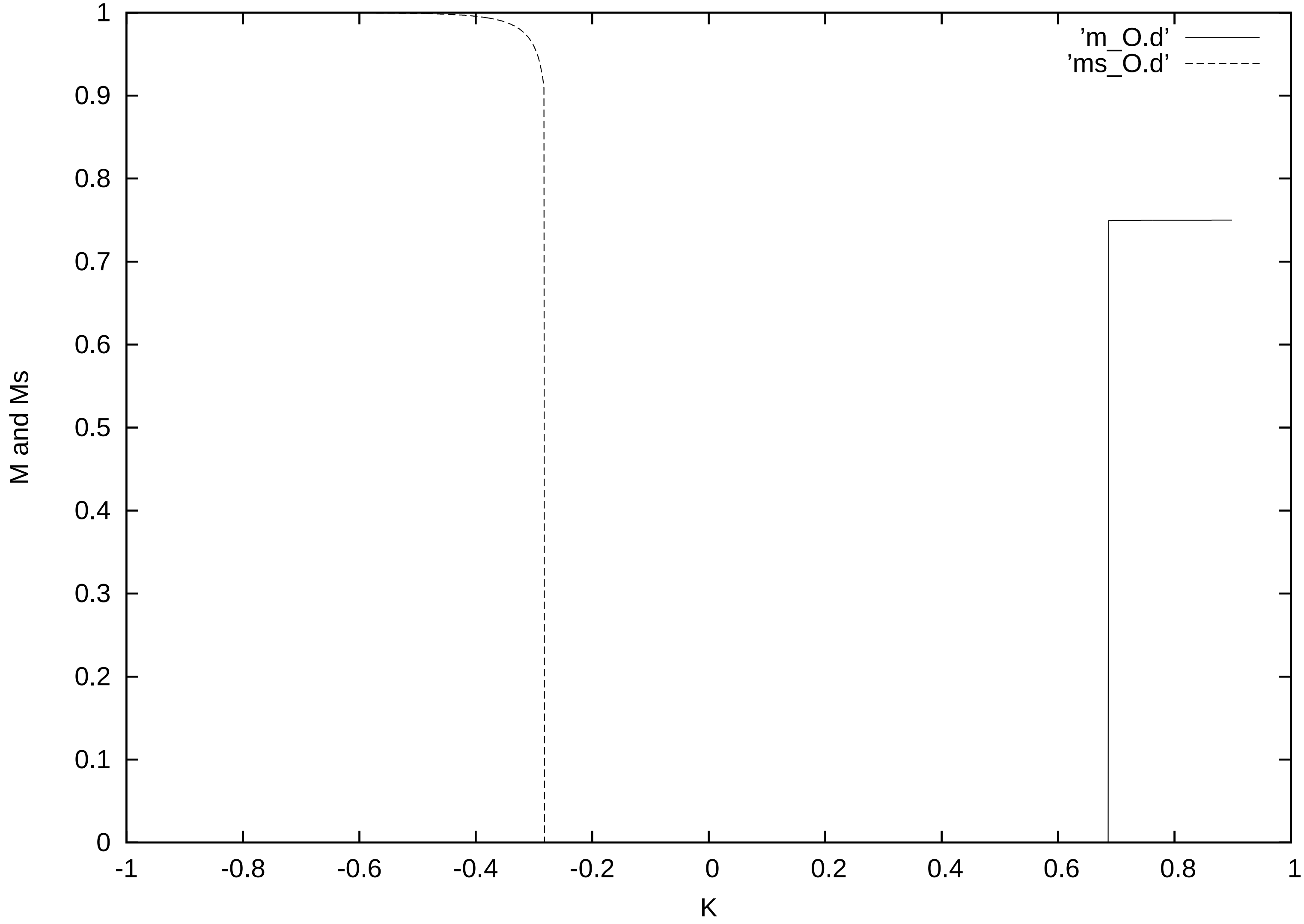
$$y = \left(\frac{4x^2 + 9x + 6y^{\frac{1}{2}} + 3}{4x^2 + 9x + 6y^{-\frac{1}{2}} + 3} \right)^3 \left(\frac{x^3 + 12x^2 + 18x + 7y^{\frac{1}{2}} + y^{-\frac{1}{2}} + 4}{x^3 + 12x^2 + 18x + 7y^{-\frac{1}{2}} + y^{\frac{1}{2}} + 4} \right)^2$$

$$x = \frac{1}{u} \times \frac{x^3 + 8x^2 + 9x + y^{-\frac{1}{2}} + y^{\frac{1}{2}} + 1}{(4x^2 + 9x + 6y^{\frac{1}{2}} + 3)^{\frac{1}{2}} (4x^2 + 9x + 6y^{-\frac{1}{2}} + 3)^{\frac{1}{2}}} \tag{20}$$

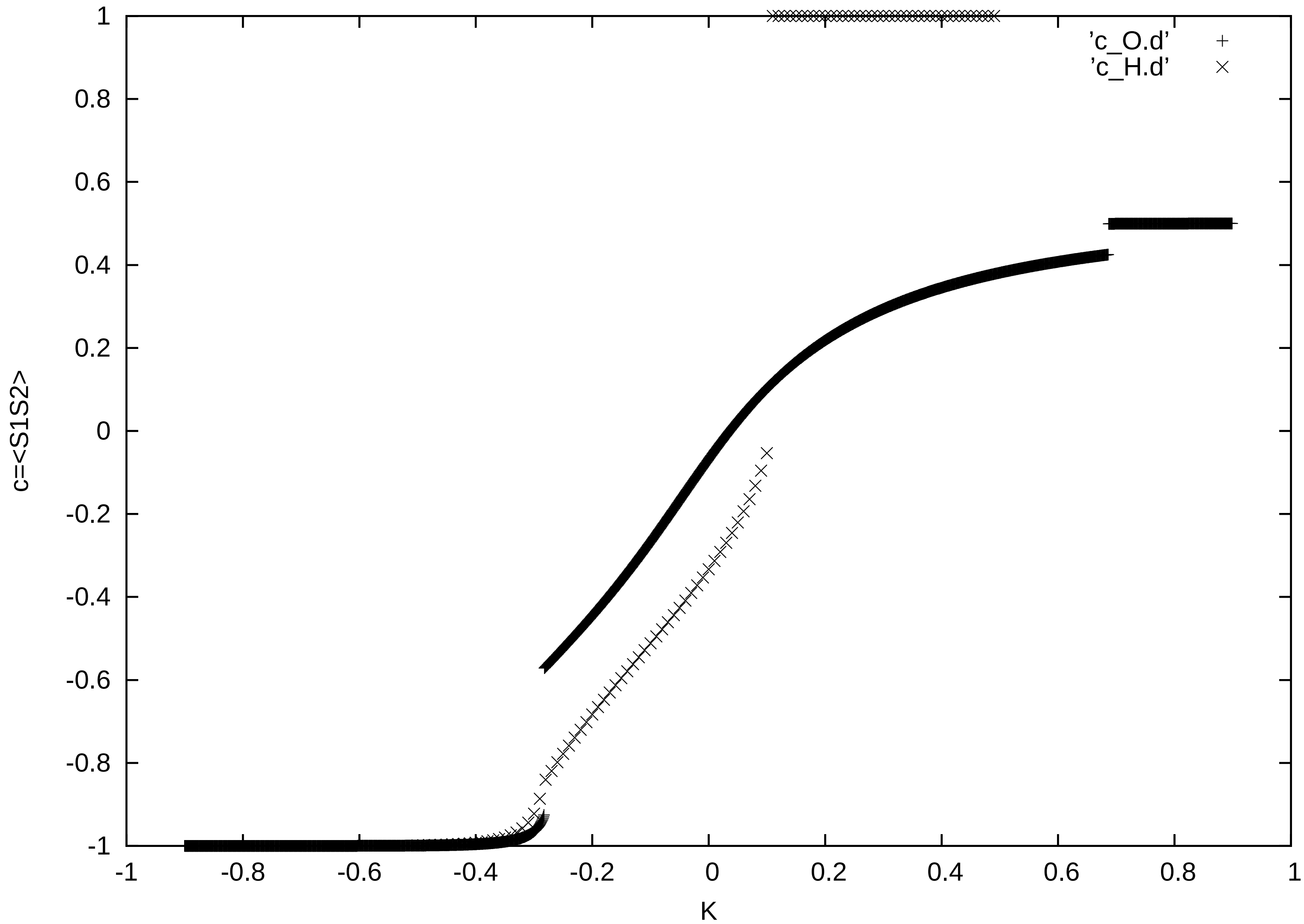
Disorderd Phase $(y = 1)$

$$x = \frac{1}{u} \frac{3 + 9x + 8x^2 + x^3}{9 + 9x + 4x^2}$$

Phase Transitions

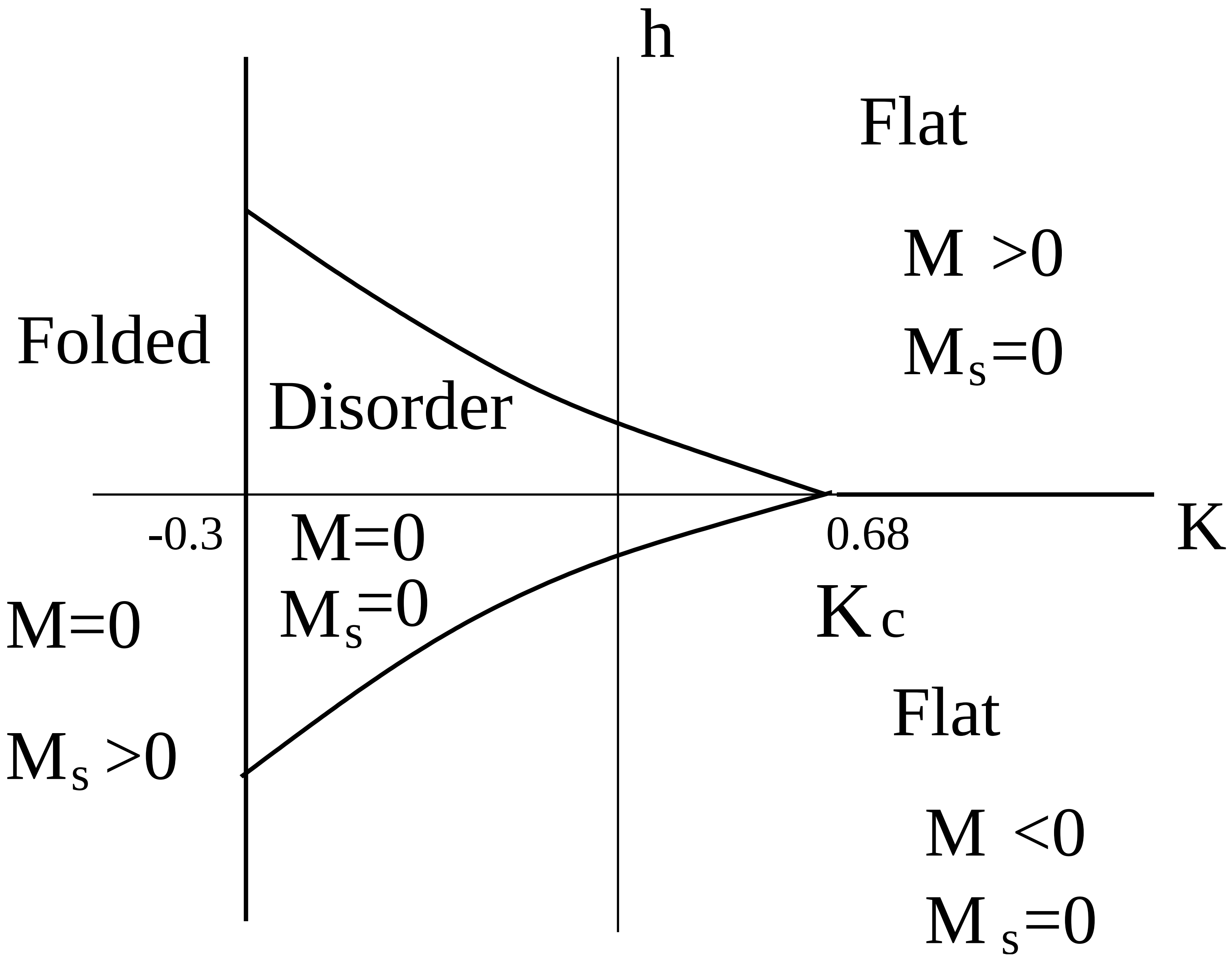


$$c = < S_1 S_2 >$$

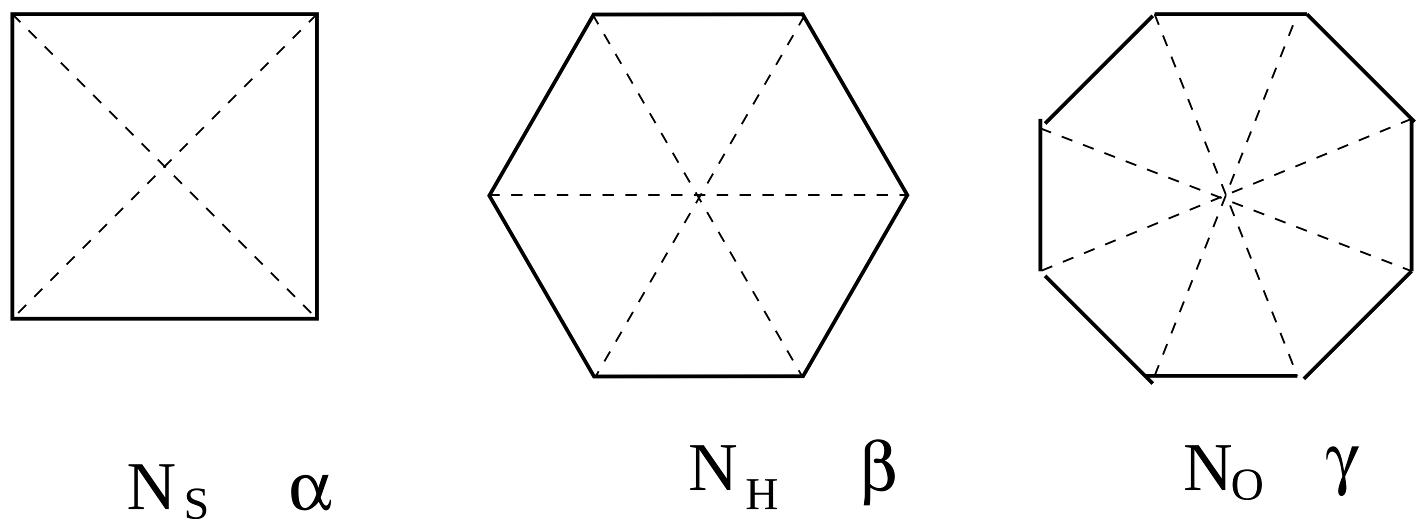


'c_O.d'
'c_H.d'
c=<S1S2>
K
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Phase Diagram



General Case ($\alpha = \gamma$)



- 4-valent Vertex $\rightarrow$ (Strong) Anti-Ferro.
- 6-valent Vertex $\rightarrow$ Not-Frustrated, (Weak) Anti-Ferro.
- 8-valent Vertex $\rightarrow$ Frustrated

Algebraic Relations for the Disordered Phase ($M = M_S = 0$)

Introducing x (Fugacity per Fold) as

$$x = \frac{1}{u}\frac{P_2(1,-1)}{P_2(1,1)}$$

All weights are written with Z_0^S, Z_0^H, Z_0^O and x as

$$\begin{aligned} &Z_1^S = Z_0^S x \\ &Z_1^H = Z_0^H x \quad Z_2^H = Z_0^H x^2 \quad Z_3^H = Z_0^H x^3 \\ &Z_1^O = Z_0^O \quad Z_2^O = Z_0^O x \quad Z_3^O = Z_0^O x \quad Z_4^O = Z_0^O x \\ &Z_5^O = Z_0^O x^2 \quad Z_6^O = Z_0^O x^2 \quad Z_7^O = Z_0^O x^3 \end{aligned} \tag{21}$$

with

$$Z_0^S(4+2x) = 1 \quad Z_0^H(2+6x+12x^2+2x^3) = 1 \quad Z_0^O(24+36x+24x^2+2) = 1$$

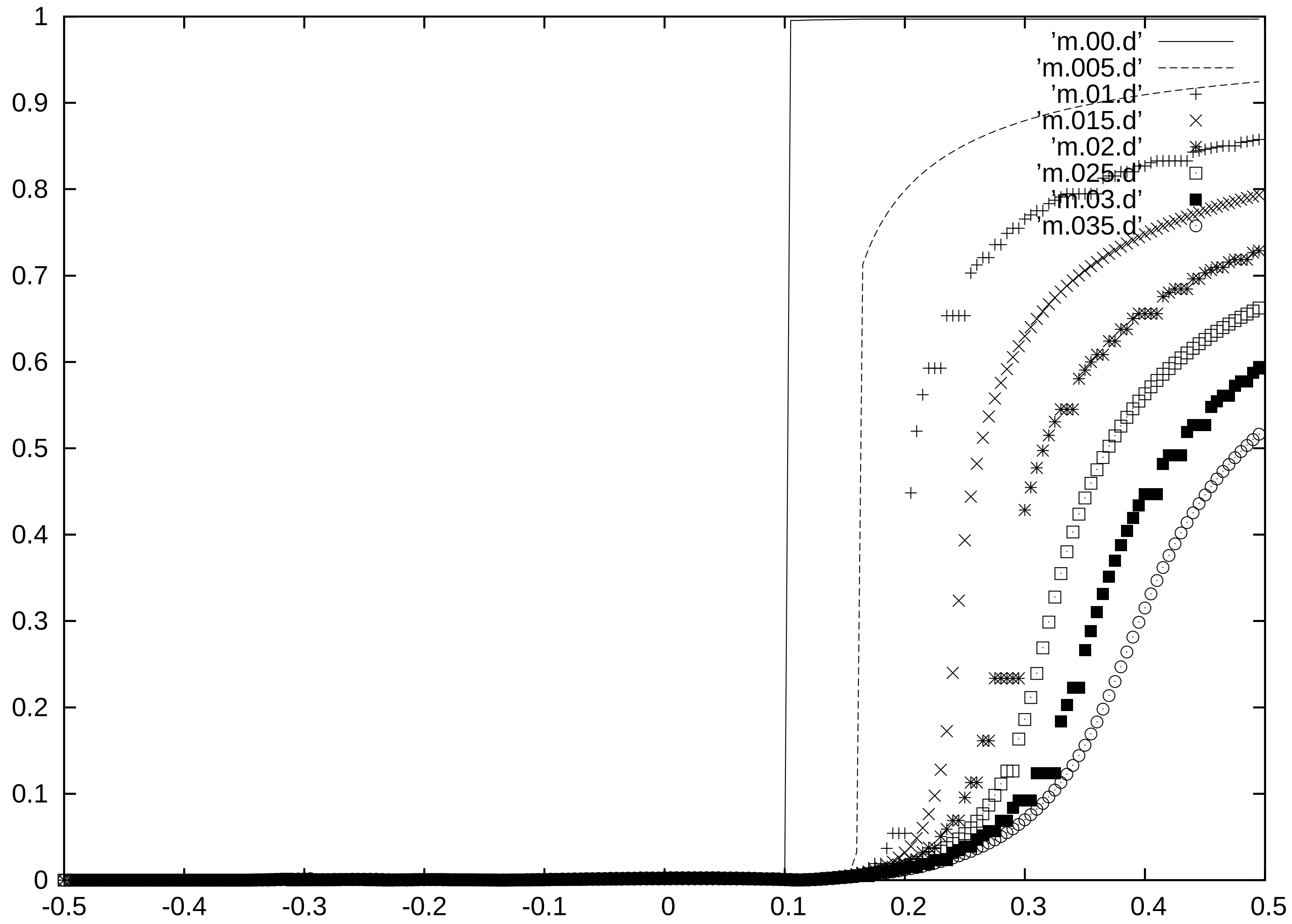
From

$$P_2(1,2) = \frac{2}{3}\alpha(\Gamma)\mathbf{STr_{3,4}}P_4(S_1,S_2,S_3,S_4)$$
$$+ \beta(\Gamma)\mathbf{STr_{3,4,5,6}}P_6(S_1,S_2,S_3,S_4,S_5,S_6)$$
$$+ \frac{4}{3}\gamma(\Gamma)\mathbf{STr_{3,4,5,6,7,8}}P_8(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8) \quad (22)$$

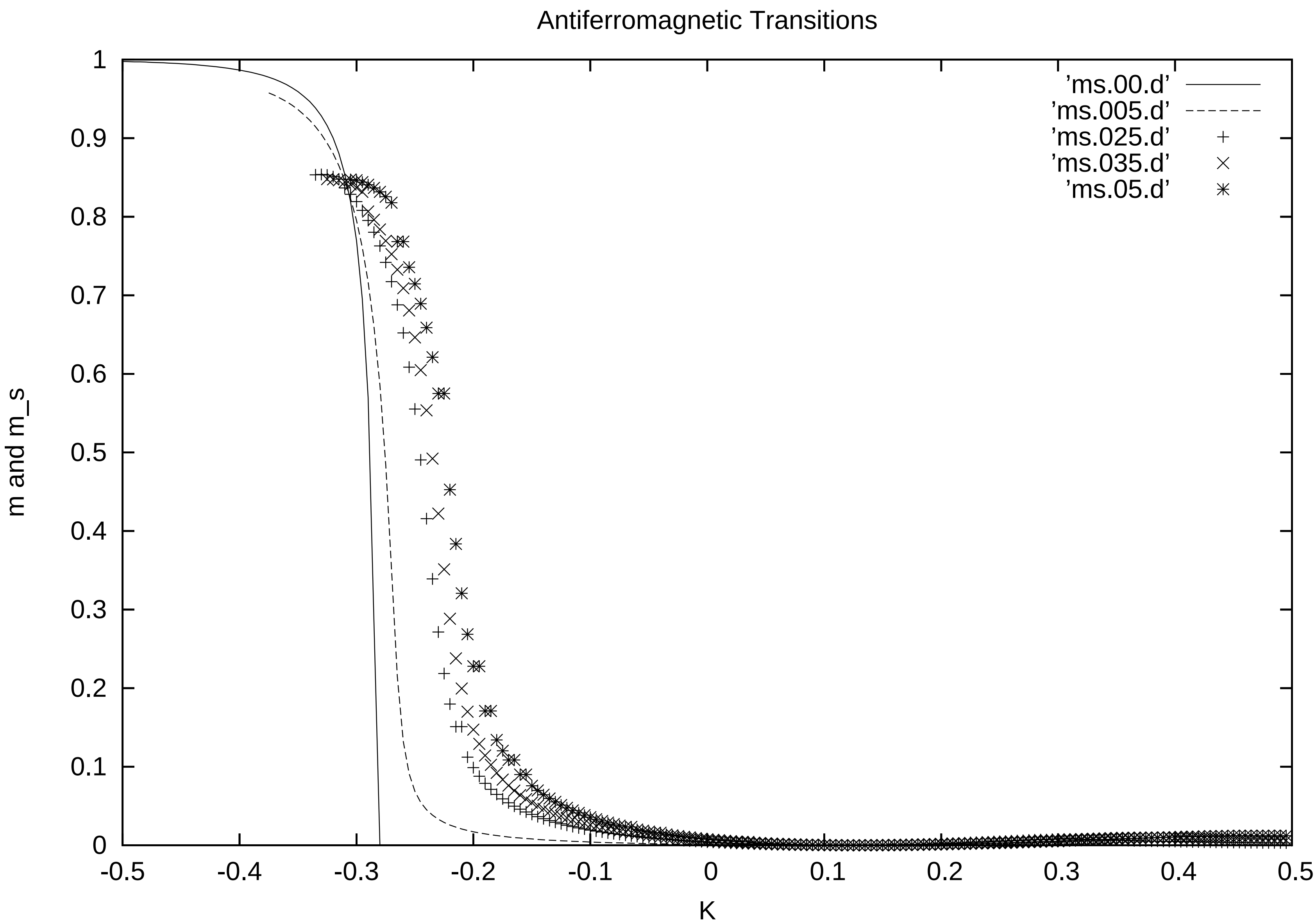
We obtain

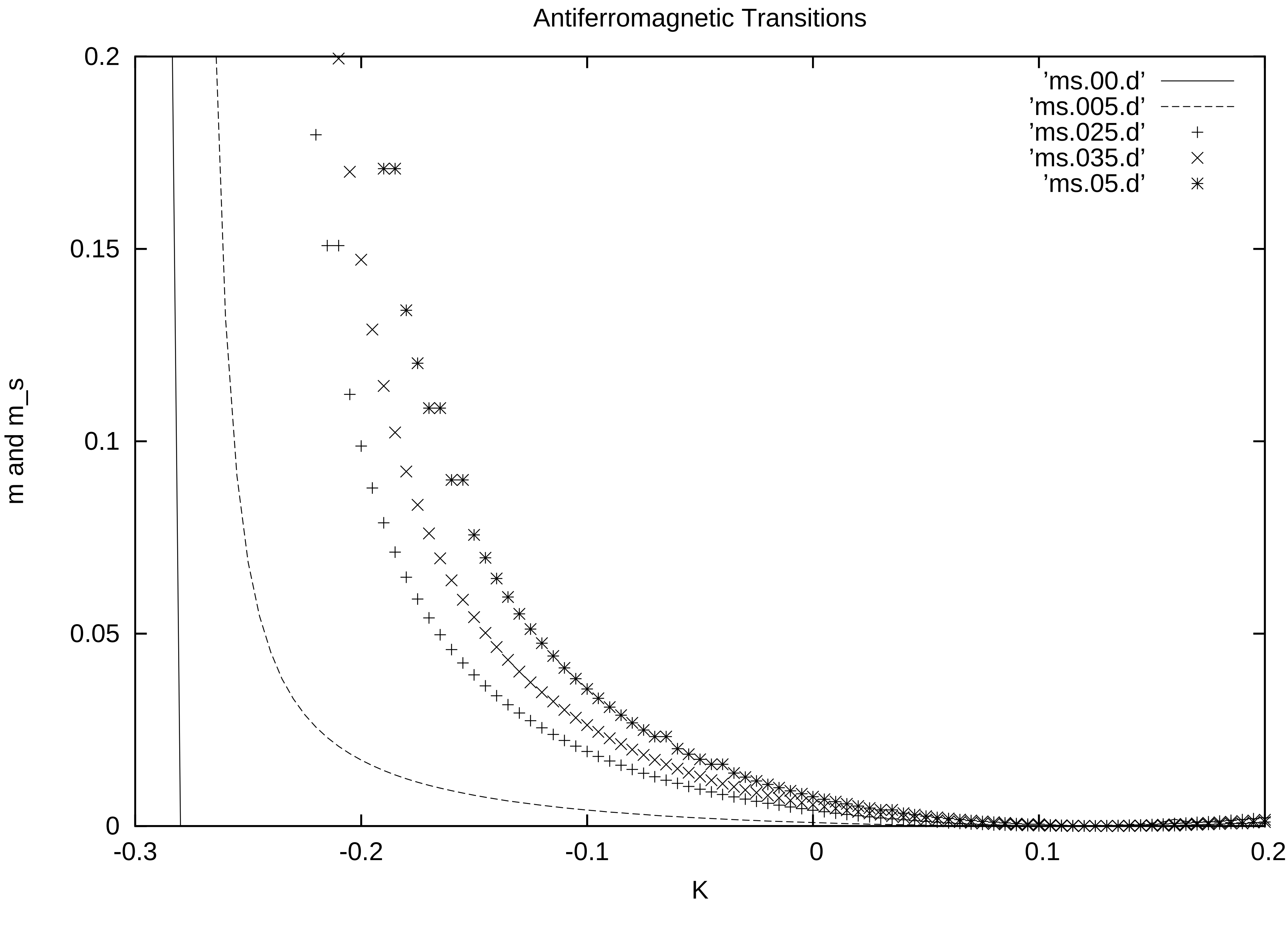
$$x = \frac{1}{u}\frac{\frac{2}{3}\alpha\frac{1+x}{4+2x} + \beta\frac{x+4x^2+x^3}{2+6x+12x^2+2x^3} + \frac{4}{3}\alpha\frac{3+9x+8x^2+x^3}{24+36x+24x^2+2x^3}}{\frac{2}{3}\alpha\frac{1}{4+2x} + \beta\frac{1+2x+2x^2}{2+6x+12x^2+2x^3} + \frac{4}{3}\alpha\frac{9+9x+4x^2}{24+36x+24x^2+2x^3}} \quad (23)$$

Ferromagnetic Phase Transitions



Antiferromagnetic Phase Transitions




Antiferromagnetic Transitions
'ms.00.d'
'ms.005.d'
'ms.025.d'
'ms.035.d'
'ms.05.d'
m and m_s
K
0.2
0.15
0.1
0.05
0
-0.3
-0.2
-0.1
0
0.1
0.2

Phase Diagram in (K, α)-Plane