Econophysics Colloquium2006@I.C.U.

Correlation Structures of CDO Pricing Models

How to understand "Dependency Structures".

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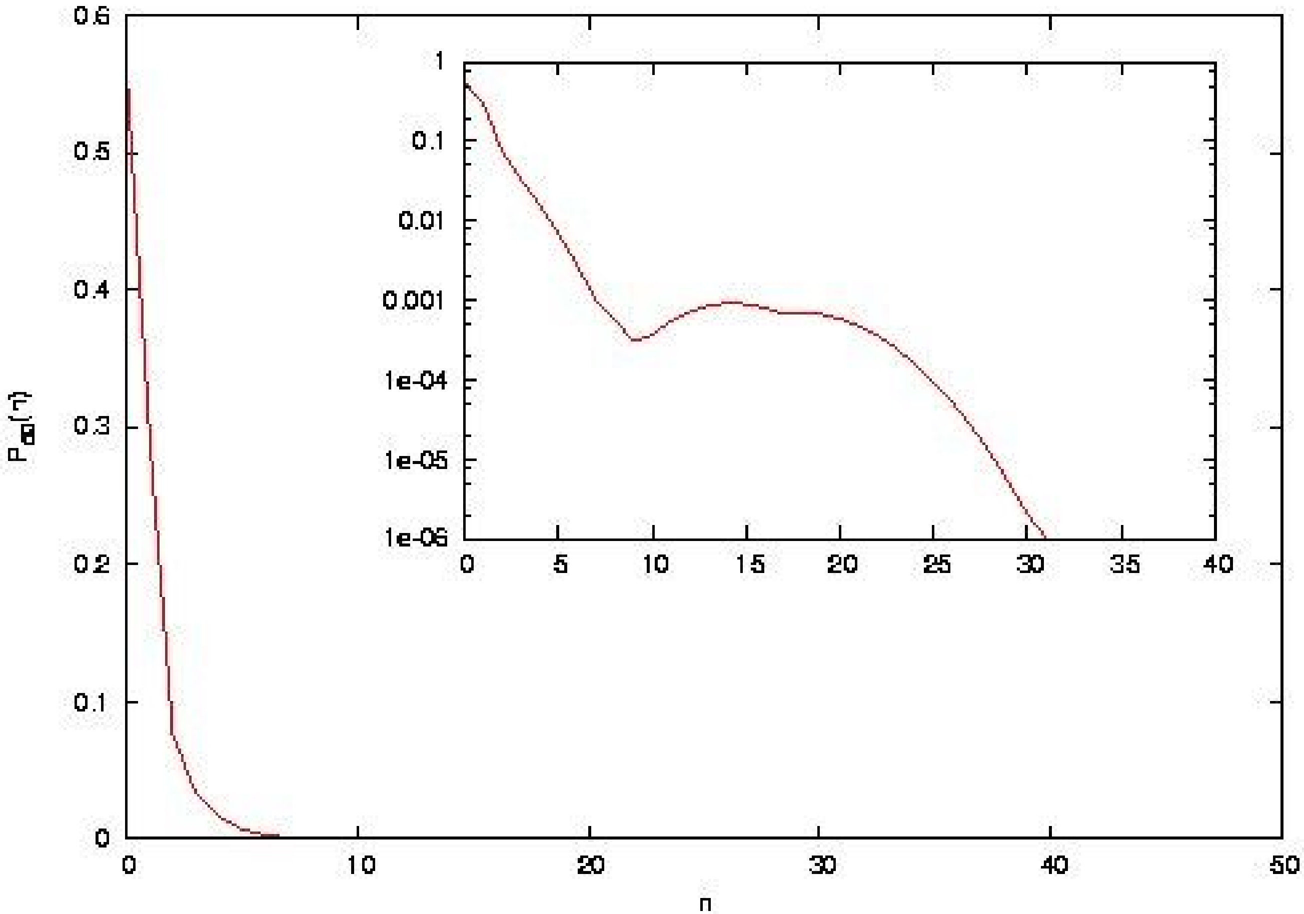
This Talk is mainly based on

- **Correlated Binomial Models and Correlation Structures, M.Hisakado, K.Kitsukawa and S.Mori,J.Phys.A39(2006)1.**
- **Default Distribution and Credit Market Implications, S.Mori, K.Kitsukawa and M.Hisakado,arXiv:physics/0609093**

Credit Market Implications

$P_{50}(n)$



What Information we get ?

What is Credit Market Implication ?

Credit Market Implications

Default Probability p

$$p = \frac{<n>}{N} \quad \Rightarrow \quad p = 1.65\%$$

Default Correlation ρ

$$\rho = \frac{<n^2 - n> / N(N-1) - p^2}{p(1-p)}$$

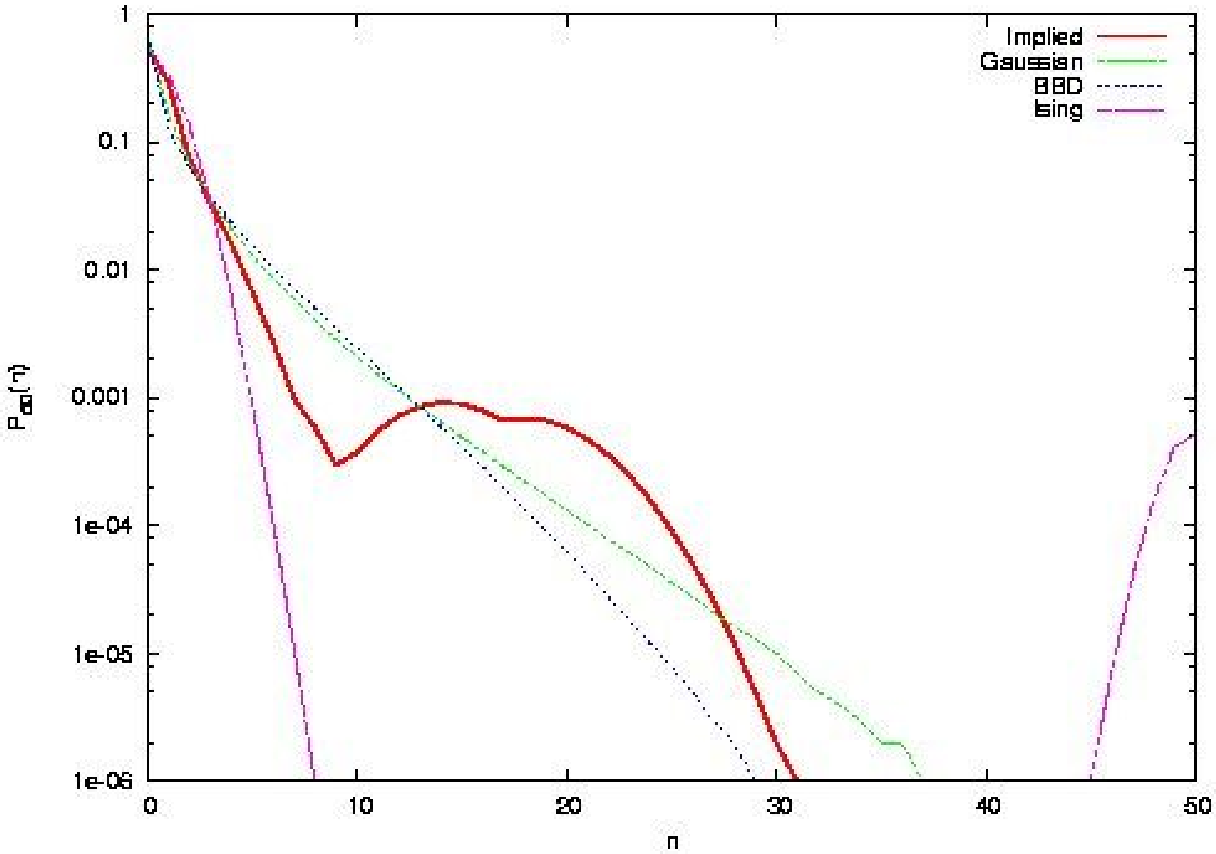
$$\Downarrow$$

$$\rho = 6.55\%$$

More Information ?

Dependency Structures ?

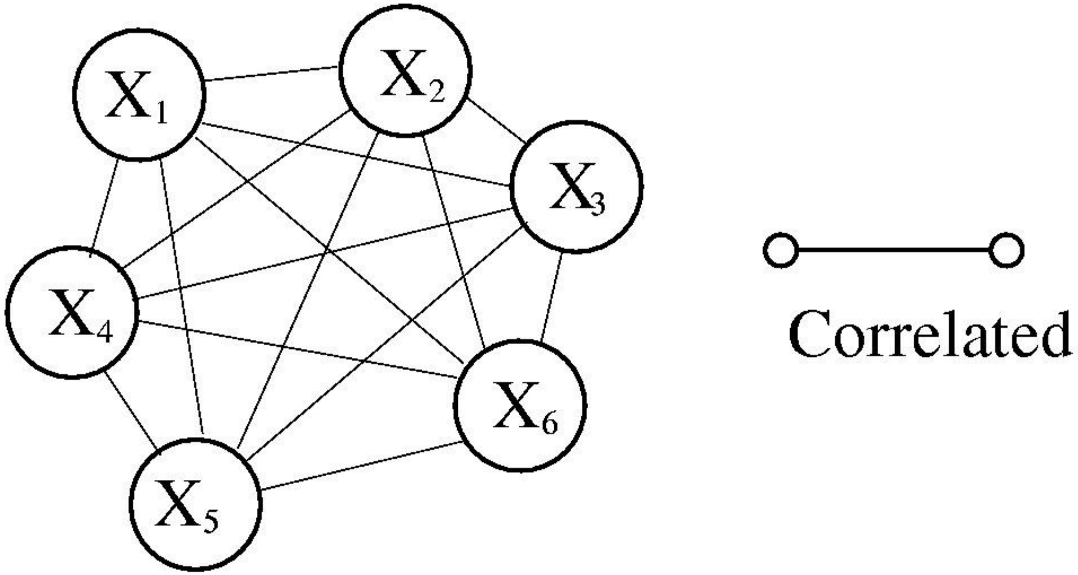
$P_{50}(n)$ $\quad\quad$ $p = 1.65\%$ $\quad$ $\rho = 6.55\%$



Why They are different ?

How to Understand their Differences.

Exchangeable and Not Independent



Exchangeable and not independent .

Bernoulli Random Variables

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$$i = 1, 2, \cdots, N$$

Exchangeable

$$X_1, X_2, \cdots, X_N \leftrightarrow X_{j_1}, X_{j_2}, \cdots, X_{j_N}$$

$$\begin{aligned} & \text{Prob}(X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N) \\ = \; & \text{Prob}(X_{j_1} = x_1, X_{j_2} = x_2, \cdots, X_{j_N} = x_N) \end{aligned}$$

How to Construct Joint Probability Function

$$P(x_1, x_2, \cdots, x_N)$$

- How to Construct ?
- Correlation Structure ρ_{ij}, p_{ij}
- Given Distribution

  → Correlation Structure $P_N(n) \rightarrow \rho_{ij}, p_{ij}$

With it, we read Credit Market Implications.

Construction of Correlated Binomial Model (CBM)

$$X_1, X_2, \cdots, X_N$$

$$\Pi_{ij} = \prod_{i'=1}^{i} X_{i'} \prod_{j'=i+1}^{i+j} (1 - X_{j'})$$

$$X_{ij} = < \Pi_{ij} >$$

Joint Probabilities
Observables

$$X_{00} = 1 \quad X_{10} = p \quad X_{01} = 1 - p = q$$

Construction of CBM

Conditional Probabilities

$$p_{ij} =< X_{i+j+1} | \Pi_{ij} = 1 >= \frac{X_{i+1j}}{X_{ij}}$$

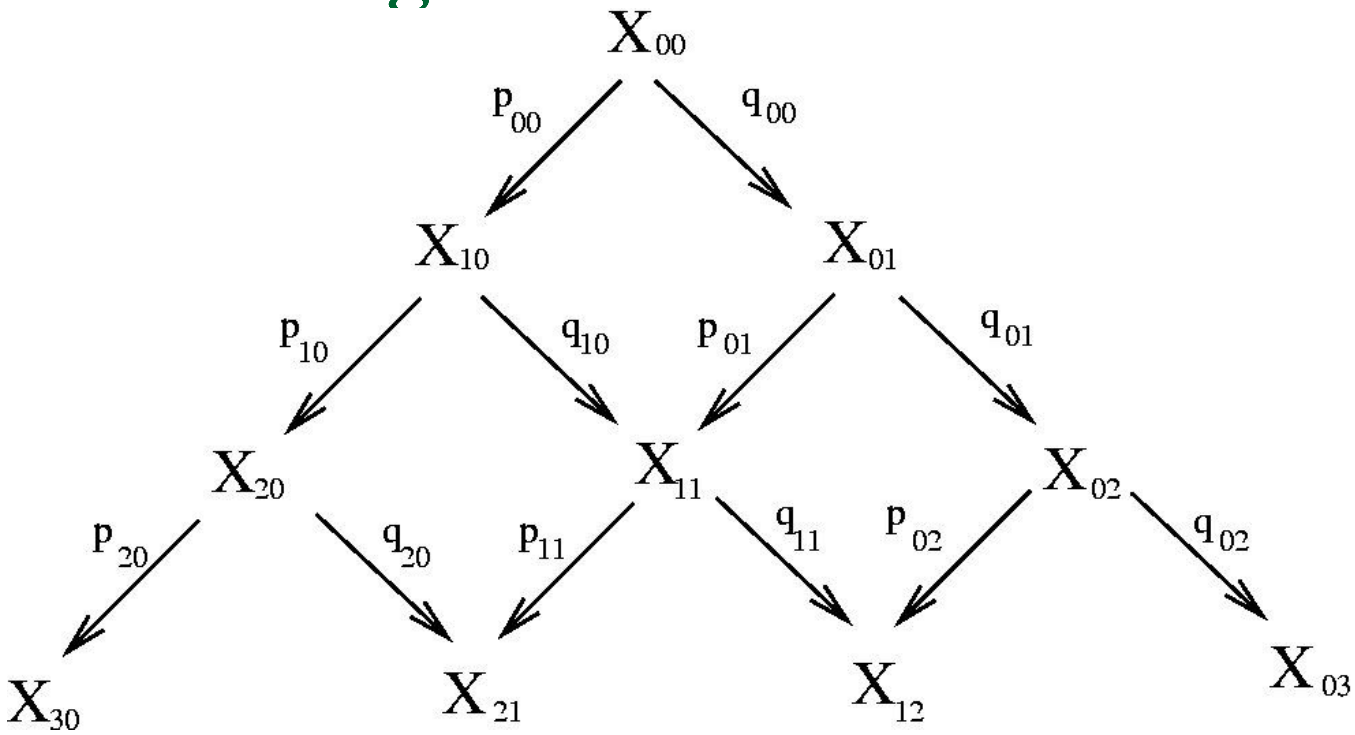
$$q_{ij} =< 1 - X_{i+j+1} | \Pi_{ij} = 1 >= \frac{X_{ij+1}}{X_{ij}}$$

$$p_{ij} + q_{ij} = 1$$

Probability Conservation Condition

$$p_{00} = p \qquad q_{00} = 1 - p$$

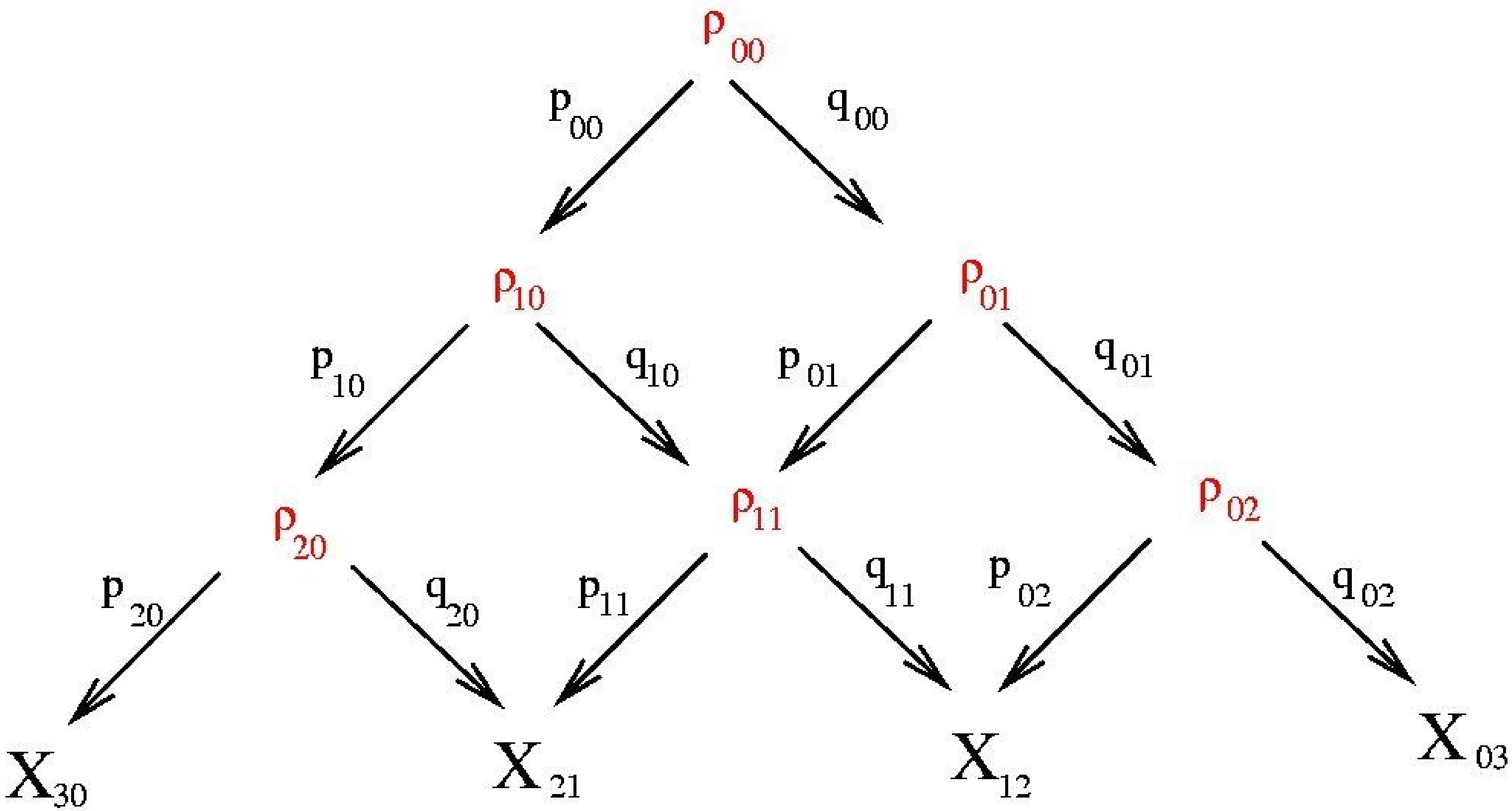
Pascal's Triangle



$$X_{nN-n} = \prod_{(00)\rightarrow(nN-n)}^{\text{Any Path}} p_{ij} \text{ and } q_{ij}$$

Must not depend on the path choice !

Pascal's Triangle



$$\mathrm{Corr}(X_{i+1}, X_{i+2} | \Pi_{ij} = 1) = \rho_{ij}$$

Commutation Relation holds.

Recursive Relations of C.B.M.

$$p_{ij} + q_{ij} = 1$$

↓

Probability Conservation Condition

ρ_{ij} and p_{ij} **Must satisfy**

$$p_{i+1j} = p_{ij} + (1 - p_{ij})\rho_{ij}$$

$$p_{ij+1} = p_{ij} - p_{ij}\rho_{ij}$$

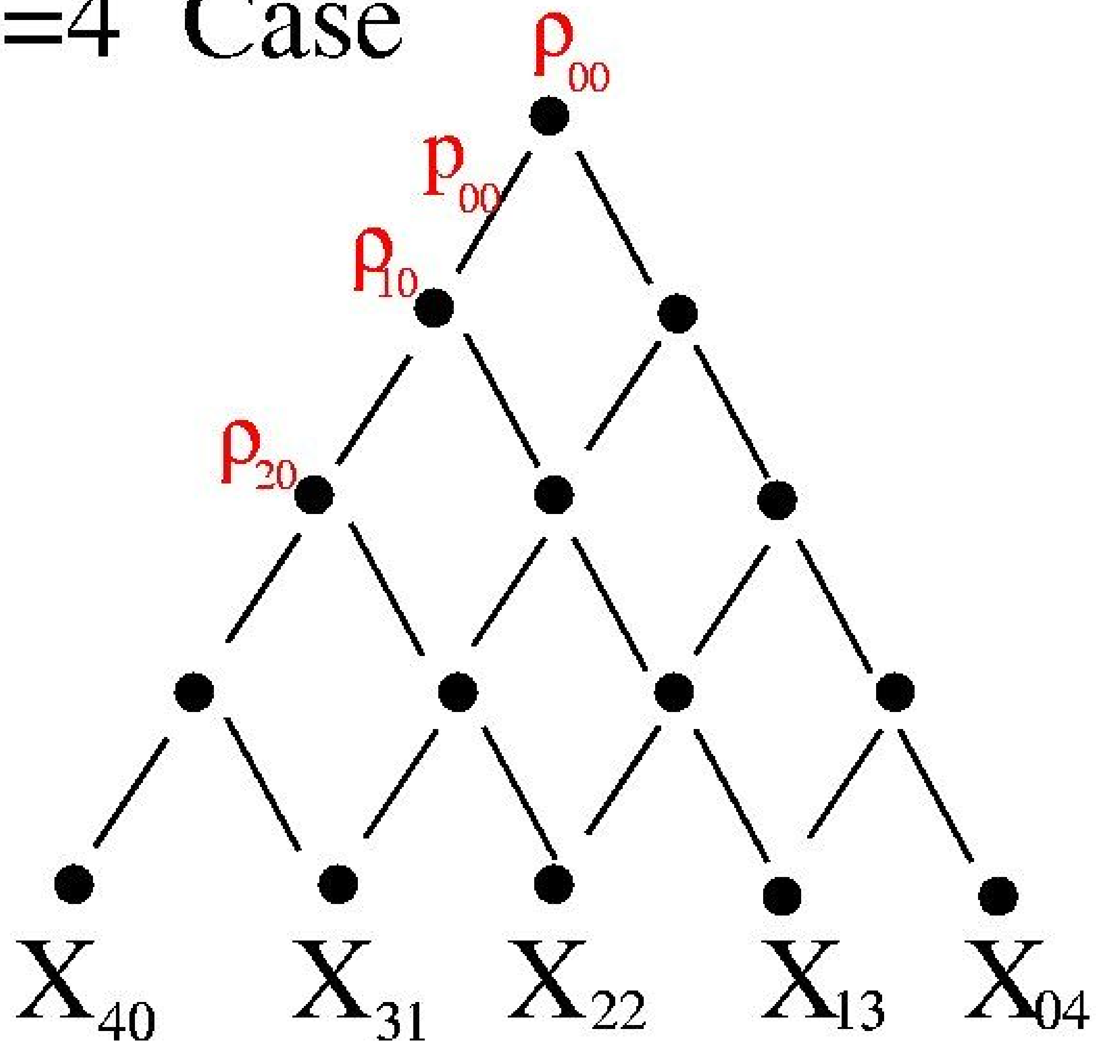
Recursive Relations

$$p_{i-1j} - p_{ij-1}$$

$$= -(1 - p_{i-1j})\rho_{i-1j} - p_{ij-1}\rho_{ij-1}$$

N Degrees of Freedom

N=4 Case



ρ_{00}

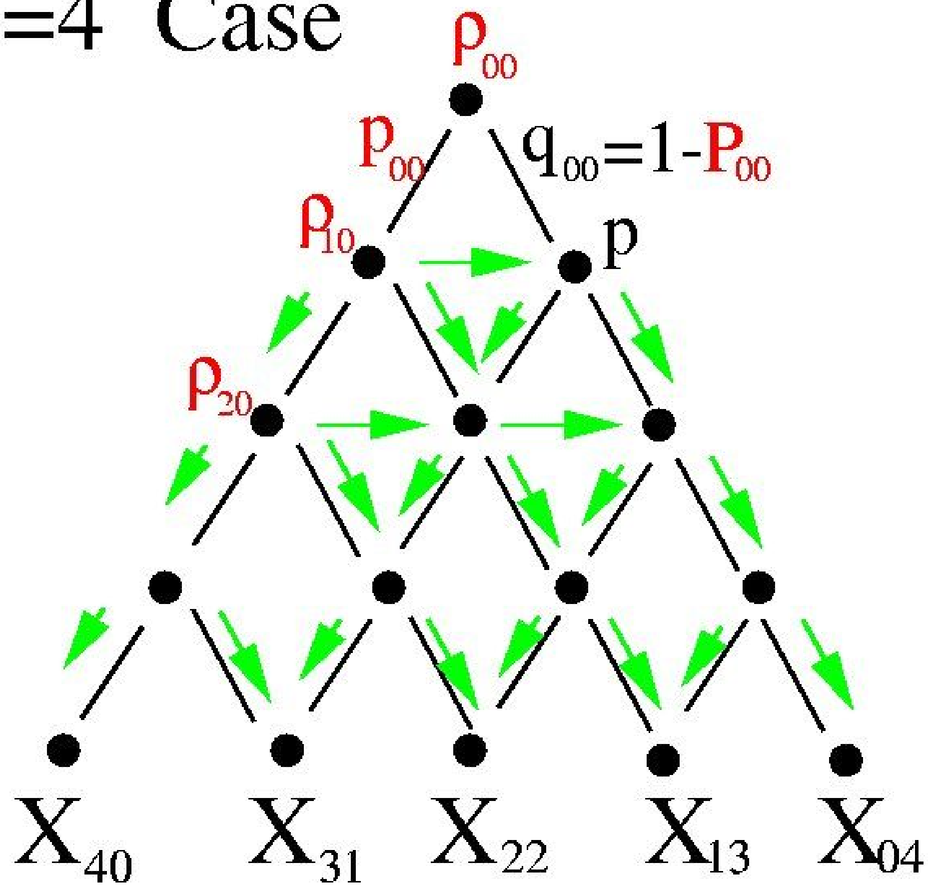
ρ_{00} ρ_{10} ρ_{20}

Independent

4 parameters

Construction of C.B.M.

N=4 Case



P_{00}

ρ_{00} ρ_{10} ρ_{20}

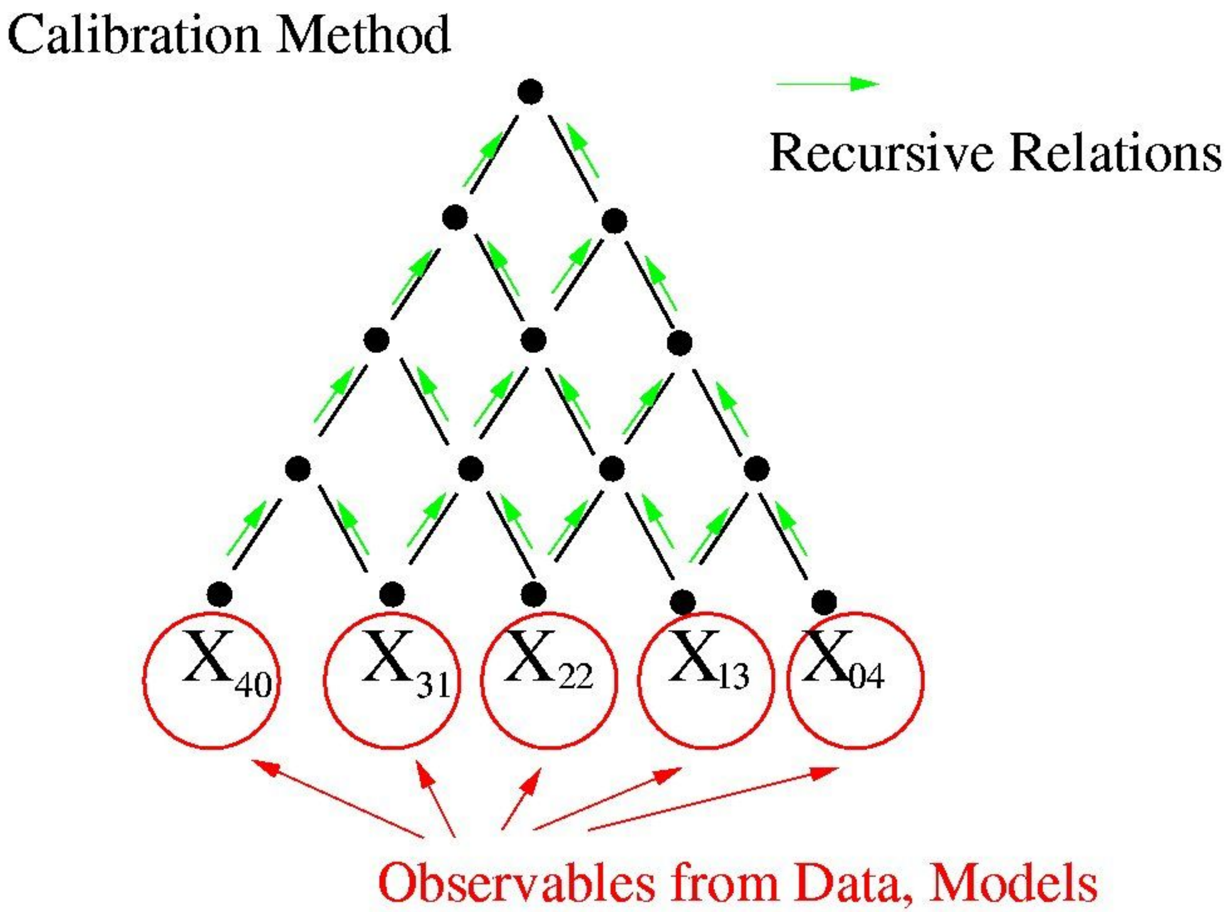
Independent

4 parameters

→ Recursive Relation

Calibration of Correlation Structures

$$P_N(n) \text{ or } X_{nN-n} \rightarrow p_{ij}, \rho_{ij}$$



C.B.M.s

$$P_N(n) = {}_NC_n \cdot \int f(p)p^n(1-p)^n dp$$

Beta Binomial Distribution (BBD model)

$$f(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} \quad \text{Beta Distribution}$$

$$X_{ij} = \frac{\mathrm{B}(\alpha+i,\beta+j)}{\mathrm{B}(\alpha,\beta)} \qquad p = \frac{\alpha}{\alpha+\beta} \quad \rho = \frac{1}{\alpha+\beta+1}$$

Long-range Ising Model

$$f(p') = (1-\alpha)\delta(p'-p) + \alpha\delta(p'-(1-p))$$

$$X_{ij} = (1-\alpha)p^i q^j + \alpha p^j q^i \text{ with } q = 1-p$$

Gaussian copula one-factor model

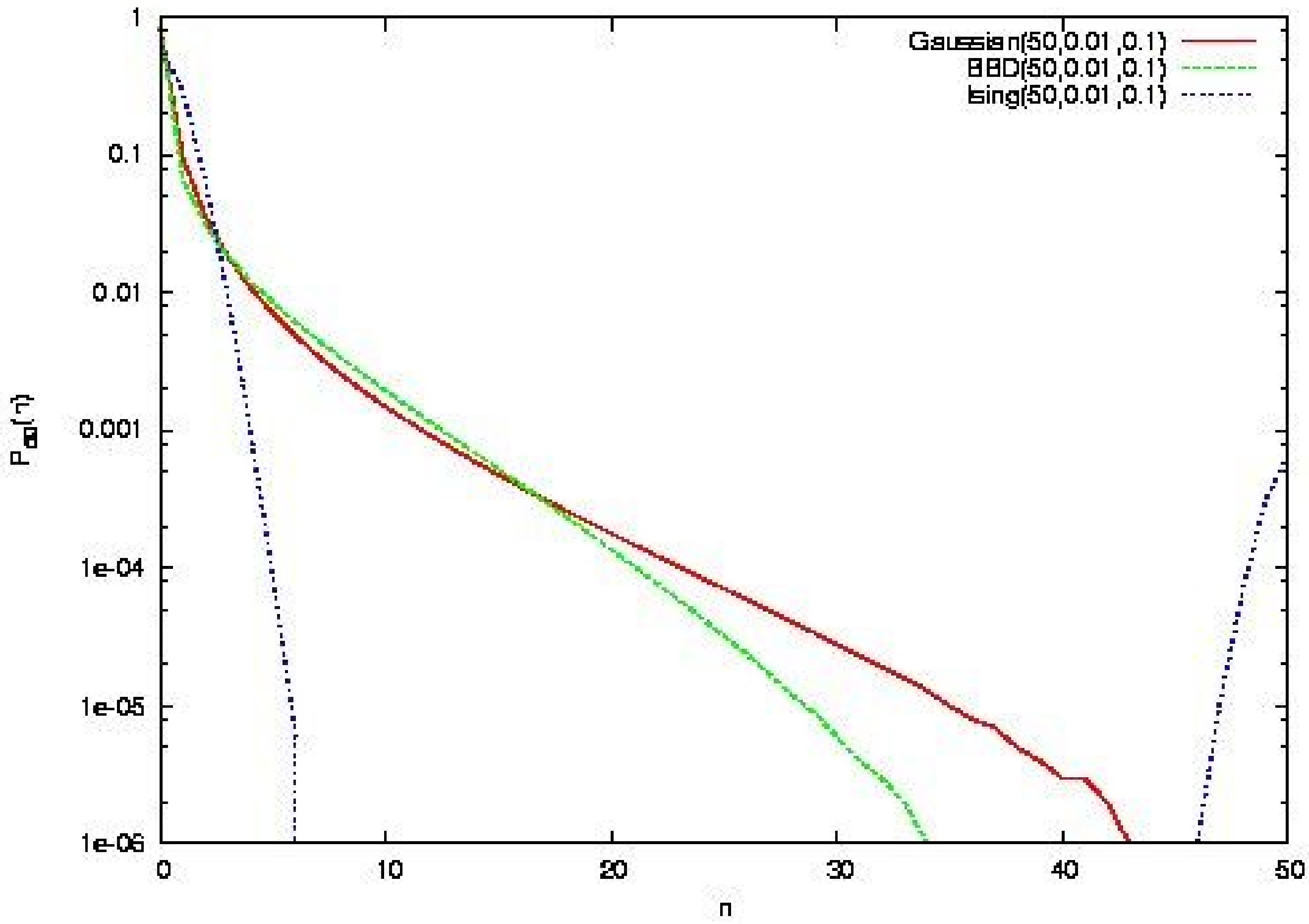
$$p(y) = \mathcal{N}[\frac{K-\sqrt{\rho}y}{\sqrt{1-\rho}}] \qquad K = \mathcal{N}^{-1}(p) \qquad y \sim N(0,1^2)$$

$$X_{ij} = < p(y)^i(1-p(y))^j >$$

C.B.M.s

$P_{50}(n)$

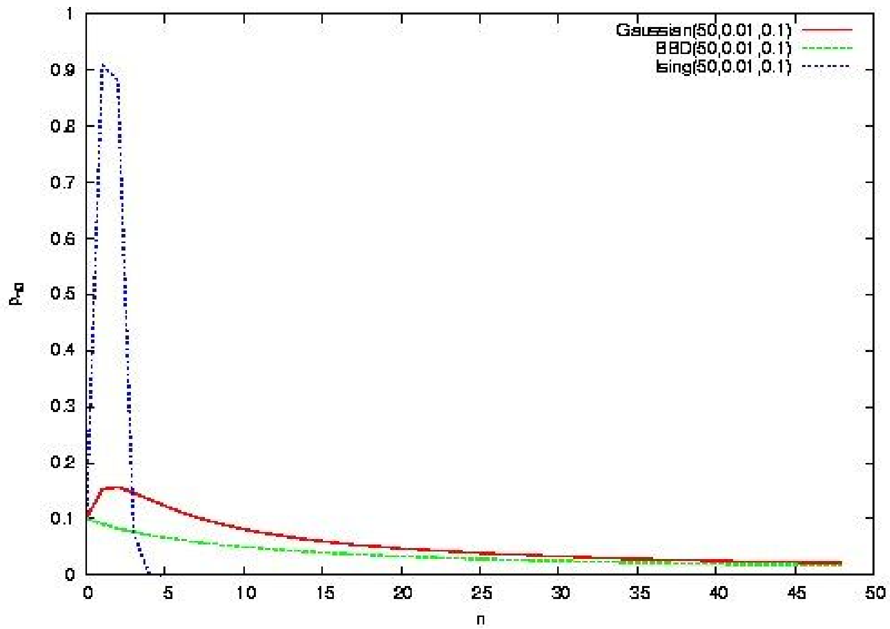
P=0.01, ρ =0.1,N=50

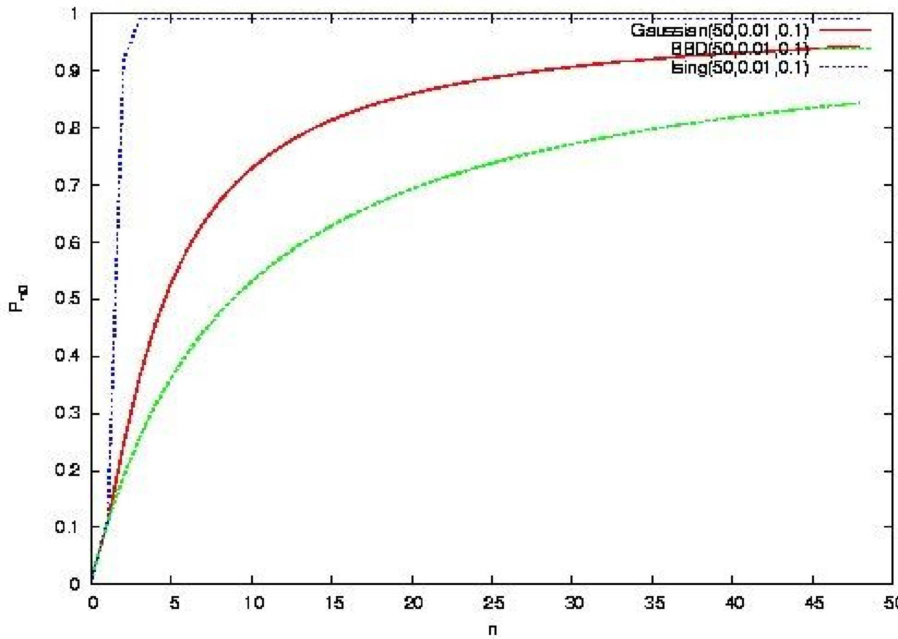


C.B.M.s

P=0.01 ρ =0.1

$\rho_{i0} = \mathrm{Corr}(X_{i+1}, X_{i+2} | \Pi_{i0} = 1)$ $\quad$ $p_{i0} = < X_{i+1} | \Pi_{i0} = 1 >$

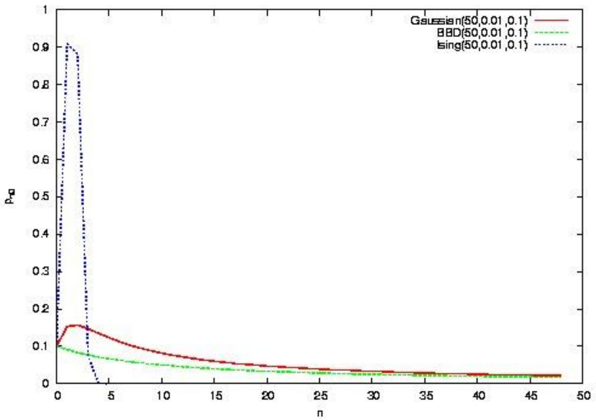


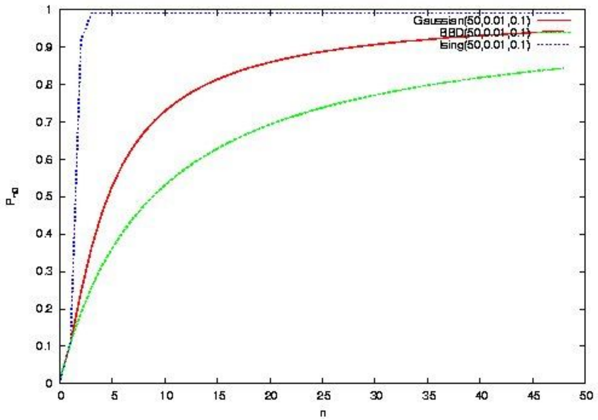


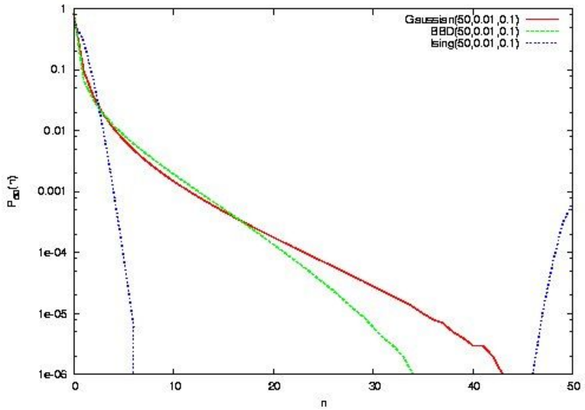
$\rho_{00} = \rho$ $\quad$ $p_{00} = p \quad p_{10} = p + (1-p)\rho$

How to Understand ?

Implications of Correlation Structures







$$\rho_{i0} \Rightarrow p_{i0} \Rightarrow P_N(n)$$

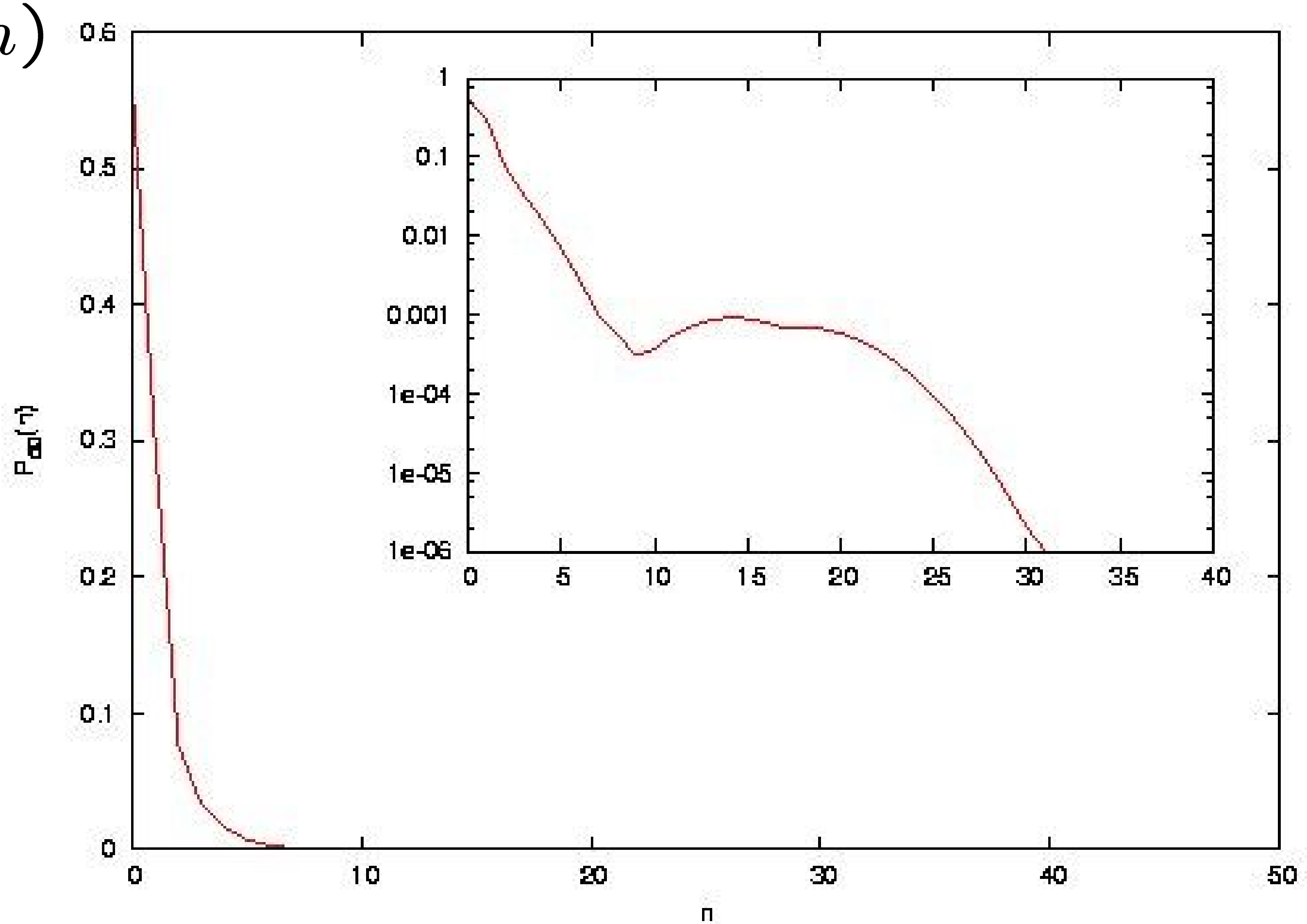
Large for small i. **Rapidly increases.** **has a fatter tail.**

Large Catastrophe.

$$p_{i+10} = p_{i0} + (1 - p_{i0})\rho_{i0}$$

How about Implied default Distribution ?

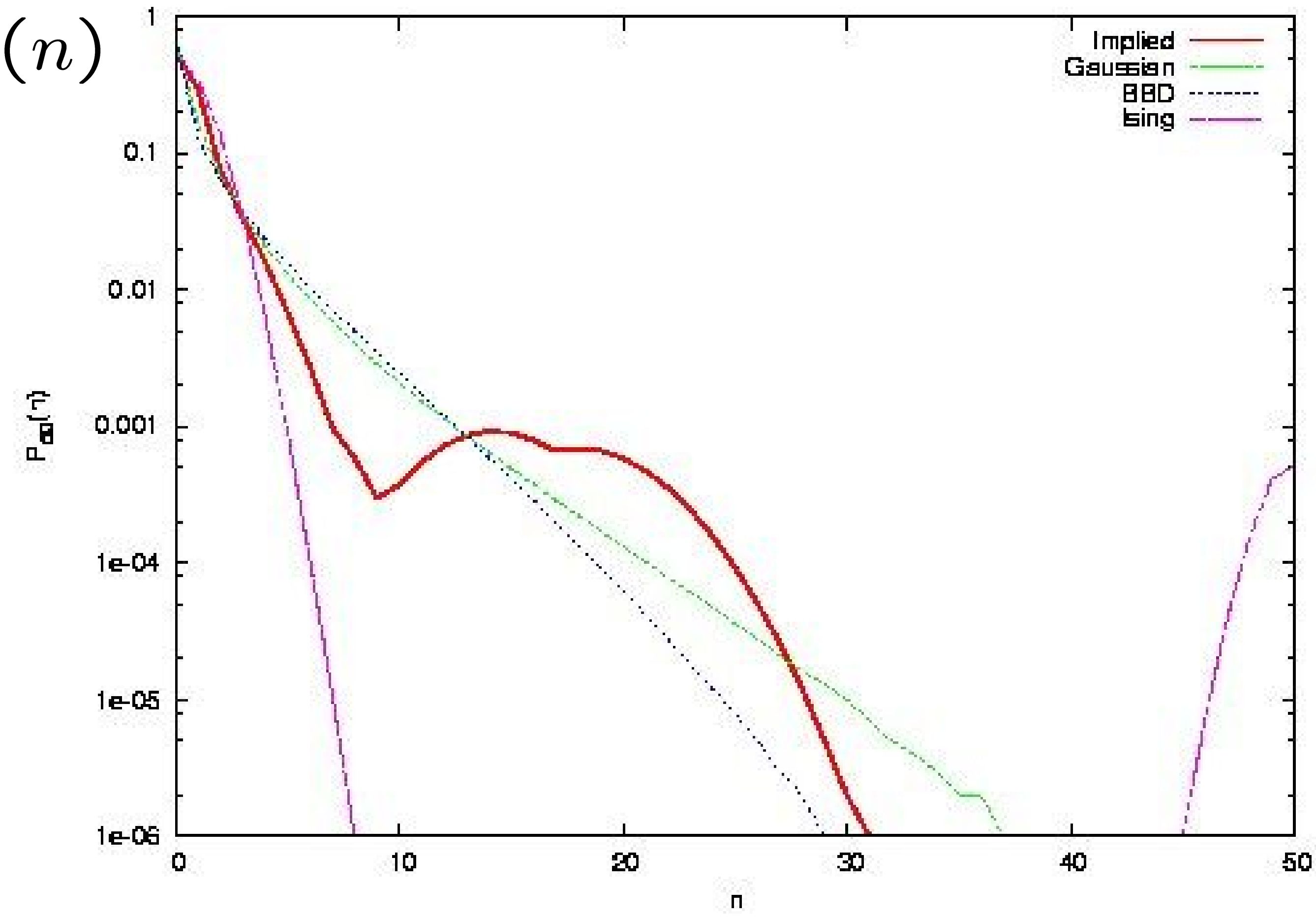
$P_{50}(n)$



Credit Market Implications

Comparison with Credit Derivative Pricing Models.

$P_{50}(n)$



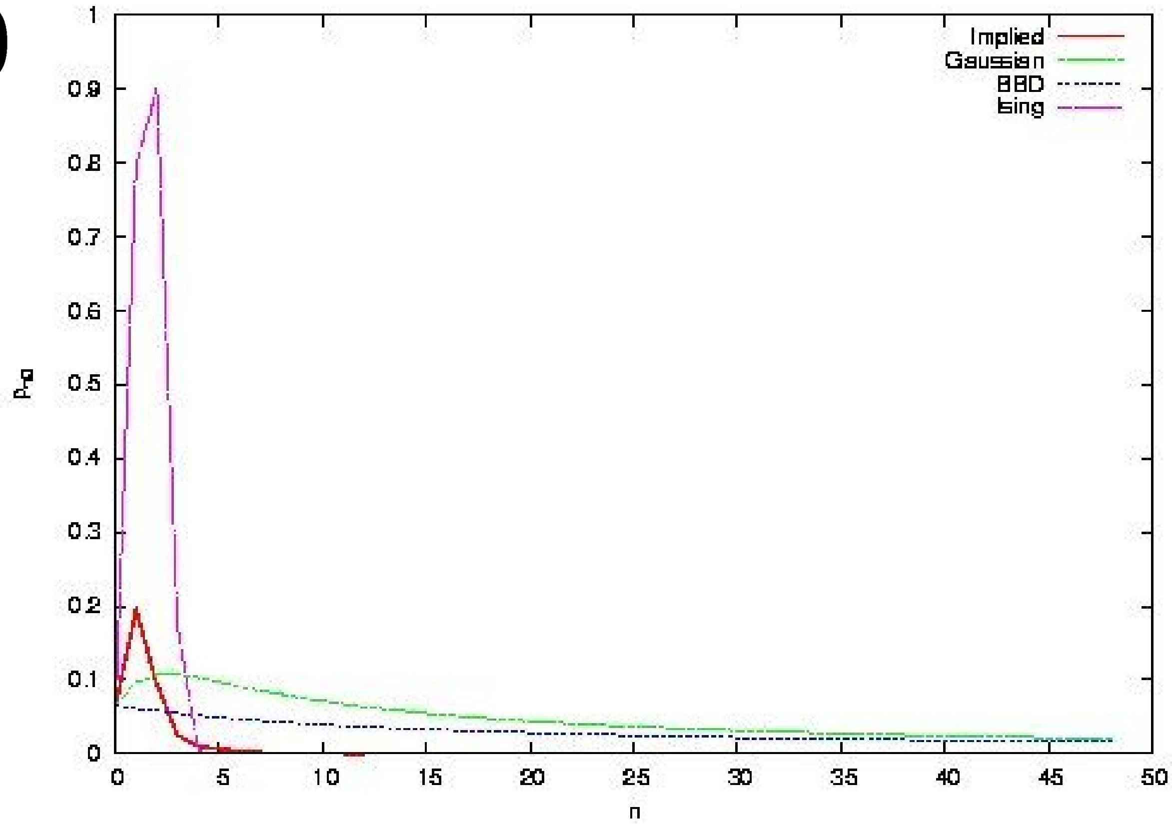
$p = 1.65\%$

$\rho = 6.55\%$

Why They are different ?

Credit Market Implications

ρ_{i0}

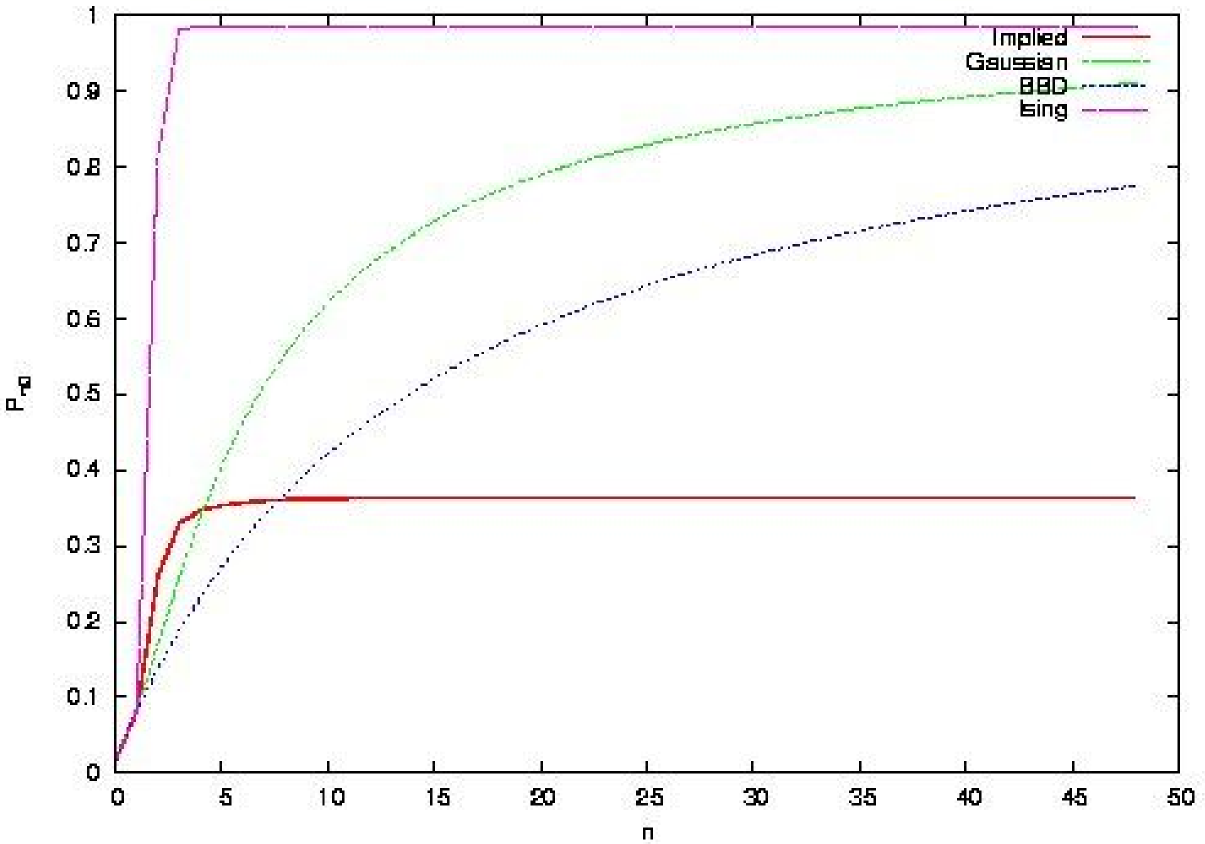


Ising and Implied look similar.

ρ_{i0} **has a peak and then rapidly decreases to zero.**

Credit Market Implications

p_{i0}



Ising and Implied look similar.

p_{i0} **starts at p and then rapidly saturates at 0.35.**

Credit Market Implications

Implied Distribution and **Ising Model** have similar properties.

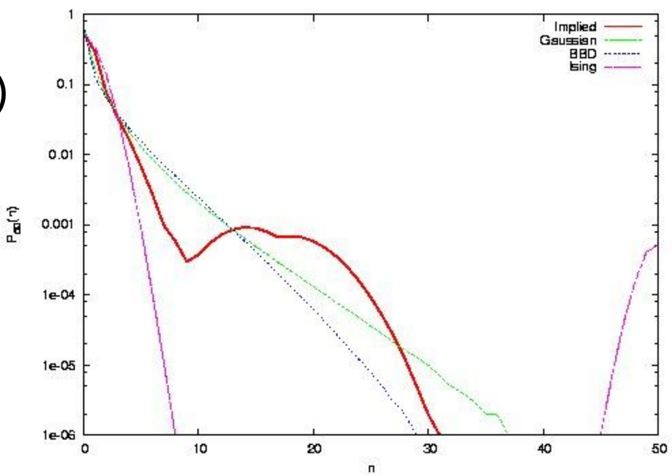
Ising = Bi(p,50) + Bi(1-p,50)

Independent Default + Big Catastrophe.

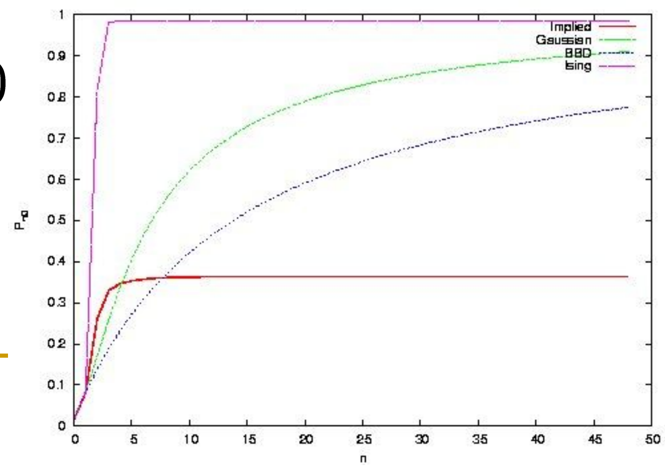
Implied Distribution = Bi(p,50)+ Bi(0.35,50)

Almost Independent Default + Medium Catastrophe.

$P_{50}(n)$



p_{i0}



Summary

- **Correlation Structure of Correlated Binomial Model.**

  Pascal's Triangle Construction.

- **Calibration Method.**

  Inverse Usage of Pascal's Triangle.

- **Implication from Correlation Structures.**

  Relation with Probability Function.

- **Implications from Implied Default Distribution.**

  Credit market (iTraxx-CJ etc) does not expect "Big Catastrophe".

  Note : It depends on the "Optimization" method.
  Entropy Maximum Principle's Implication.

Information

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- **Default Distribution and Credit Market Implications, S.Mori, K.Kitsukawa and M.Hisakado,arXiv:physics/0609093.**

Tomorrow's Poster Session

- **Infectious Default Model with Recovery and Continuous Limit A.Sakata, M.Hisakado and S.Mori, arXiv:physics/0610275.**