Critical properties of non-linear Pólya urns

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We describe a universality class of the transitions of a generalized Pólya urn by studying the critical behavior of the normalized correlation function *C*(*t*).

Non-linear Pólya urn

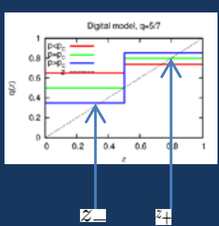
$X(t) \in \{0,1\}, t = 1,2,\cdots,$

$\Pr(X(t+1) = 1 | z(t) = z) = q(z)$ $\qquad z(t) = \frac{1}{t}\sum_{s=1}^{t} X(s)$

$$\begin{aligned} C(t) &\equiv \mathrm{Cov}(X(1), X(t+1))/\mathrm{Var}X(1) \\ &= \Pr(X(t+1)=1|X(1)=1) - \Pr(X(t+1)=1|X(1)=0) \end{aligned}$$

Digital Model

$q(z) = (1-p)\cdot q_* + p\cdot\theta(z-1/2), q_* \geq 1/2, p \in [0,1]$



$p_c = 1 - 1/2q_*$

$p \leq p_c, z_* = (1-p)q_* + p$

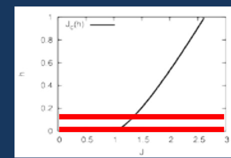
One stable state

$p > p_c, z_* = (1-p)q_* \pm p$

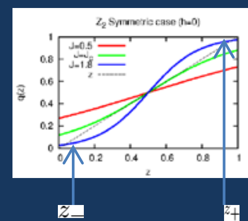
Two stable states

Ising-type Model

$q(z) = \frac{1}{2}[\tanh(J(2z-1)+h)+1], J \geq 0, h \geq 0$



$h = 0$



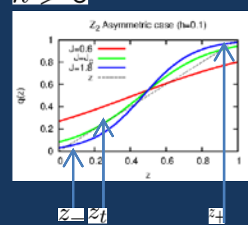
$J \leq J_c(0) = 1 : z_* = 1/2$

One stable state

$J > J_c(0) = 1 : z_* = z_\pm \neq 1/2$

Two stable states

$h > 0$



$J < J_c(h) = 1 : z_* = z_+ > 1/2$

One stable state

$J = J_c(h) = 1 : z_* = z_+, z_t$

Two stable states

$J > J_c(h) = 1 : z_* = z_\pm \neq 1/2$

Two stable states

Results

$$C(t) = b(q_*)t^{-1/2}g(t/\xi(q_*,p))$$

$b(q_*) = \sqrt{\frac{8}{\pi}}\sqrt{\frac{2q_*-1}{4q_*-1}}$ $\qquad \xi(q_*,p)^{-1} = -\log\sqrt{4(p+(1-p)(1-q_*))((1-p)q_*)}$

$$g(x) = \sqrt{4\pi x} + \frac{x^{1/2}}{2}\int_x^\infty u^{-3/2}e^{-u}du, p > p_c(q_*)$$

$$g(x) = \frac{x^{1/2}}{2}\int_x^\infty u^{-3/2}e^{-u}du, p < p_c(q_*)$$

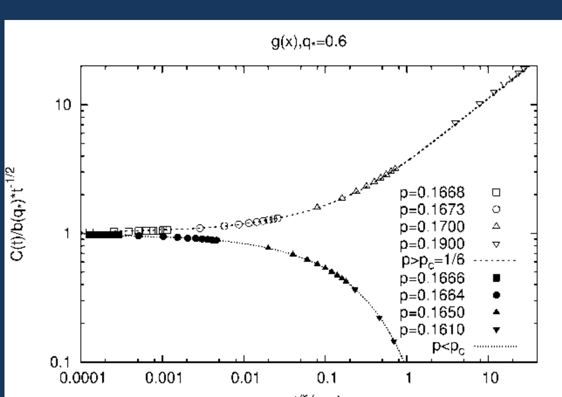
Order parameter c: Continuous transition

$$c \equiv \lim_{t\to\infty} C(t) = b(q_*)\sqrt{\frac{4\pi}{\xi(q_*,p)}}, p > p_c(q_*)$$
$$= 0, p < p_c(q_*)$$

Scaling relation

$\beta = \alpha\cdot\nu_{||}, c \propto \Delta p^\beta, C(t) \propto t^{-\alpha}, \xi \propto \Delta p^{-\nu_{||}}$

$\beta = 1, \alpha = 1/2, \nu_{||} = 2$



Results $J \neq J_c(h)$

$$C(t) \simeq c + c'\cdot t^{l-1}, l = \mathrm{Max}(q'(z_+), q'(z_-))$$

Critical Behaviors

$J = J_c(h), h > 0$ Discontinuous transition

$C(t) \simeq c + c'\cdot(\log t)^{-1}, c > 0$

$J = J_c(0) = 1$ Continuous transition

$C(t) \simeq c'\cdot(\log t)^{-1/2}, c = 0$

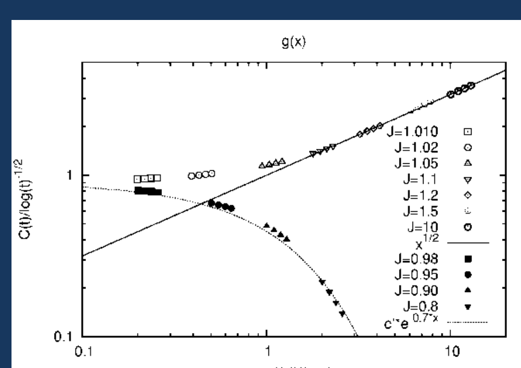
Universal function (h=0)

$C(t) \simeq (\log t)^{-1/2}g((1-l)\log t)$ $\qquad g(x) \sim x^{1/2}$

Scaling relation

$\beta = \alpha\cdot\nu_{||}, c \propto \Delta J^\beta, C(t) \propto (\log t)^{-\alpha}, \xi \equiv 1/(1-l) \propto \Delta J^{-\nu_{||}}$

$\beta = 1/2, \alpha = 1/2, \nu_{||} = 1$



Reference

Correlation function for generalized Polya urns: Finite-size scaling analysis
S.Mori and M. Hisakado, arXiv:1501.00764